Perturbed three point vortex problems

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Abstract

Making use of the knowledge of the motions of three point vortices in a planar incompressible flow, which are integrable, we investigate the motions of three-point vortices in a half-plane as perturbations of the original integrable system. Numerical investigations are carried out to show whether the perturbations will remain small or will diverge, respectively, from those corresponding to stable or unstable critical points of the original ones.

Keywords: Three point vortices; Integrable system; Perturbation; Stability

1. Introduction

The dynamics of three point vortices in a plane was analyzed and shown to be integrable by Gröbli [1] and Synge [2]. Their results were re-derived via Hamiltonian formalism by Novikov [3] and Aref [4]. Tavantzis and Ting [5] continued Synge's analysis using trilinear coordinates to obtain an integral invariant defining the integral curves or trajectories representing the variations of the configuration or triangle formed by the three vortex centers. They identified all the critical points and separatrices for a given set of strengths and initial positions of the three vortices.

Numerical studies of the motions of three vortices of equal strength in a half-plane were carried out by Knio et al. [6]. The vortex configurations can be regular or chaotic depending on their initial configuration, which was taken to be either an equilateral triangle or collinear. These results reveal the connections to the dynamics of three vortices in the plane for which the equilateral configuration corresponds to a center while the collinear configuration with one vortex in the middle corresponds to a saddle point [2]. This observation initiated the studies of the interactions of three co-axial vortex rings in a meridian plane as a perturbed planar three-vortices problem (the original problem) by Blackmore and Knio [7]. With a typical ratio of the distances between the rings to their radii as the small parameter, they proved the existence of quasi-periodic solutions when the three vortices have the same sign. Certainly, the original problem has to be elliptic. (It is called elliptic, parabolic or hyperbolic [2] when the sum of the mixed products of the strengths of vortex pairs is positive, zero, or negative.) For the original system, it was shown [2] that the equilateral configuration is stable, corresponds to a center, (unstable, a saddle point) when it is elliptic (hyperbolic). They found quasi-periodic solutions on invariant tori when the initial configuration is nearly equilateral [7]. When it is parabolic, there is a critical curve in trilinear coordinates, points at which the configurations remain stationary up to a similarity, i.e. the triangles remain similar. The equilateral triangle lies on the critical curve and divides it into two segments, representing expanding and contracting similar triangles, respectively. It was shown in [5] that the expanding similar solution is stable while the contracting one is unstable. Recently, Ting and Blackmore [8] described the bifurcations from a stationary point of contracting similar solutions along an integral curve ending at a point on the branch representing expanding triangles. Here we study the perturbed three-vortex problem when the initial configuration is nearly an equilateral triangle, which is unstable in the original problem for the hyperbolic and parabolic cases.

In the next section, we review the formulation of the original three-point vortex problem, define the symbols, summarize the results of Synge [2], and Tavantzis and Ting [5], and then formulate the perturbed problems. In Section 3, we explain our choice of the points of

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reference between the original problems and the perturbed problems, and describe the input data for the numerical investigations. The interpretation of the numerical results is then presented.

2. Background

Following the symbols of [2] and [5], we denote the strength of a vortex k_j to its circulation Γ_j by $k_j = \Gamma_j/2\pi$, $j = 1, 2, 3$, and the position of the vortex center in the complex plane by $z_j(t) = \mathrm{Re}z_j + i\mathrm{Im}z_j$.

Using the conservation of linear and angular momenta for the motion of the vortex centers, a three degree-of-freedom (six-dimensional) problem was reduced to a one degree-of-freedom (two-dimensional) problem for the variation of the sides of the triangle $\mathcal{G}$, i.e. the trajectory of point $\mathbf{R}(t)$ in space, where R_j denotes the side of the triangle facing the vertex z_j; for example, $R_3 = |z_1 - z_2|$. Synge projected the trajectory in R_j space radially onto the point $\mathbf{x}$ on the $\alpha\beta$ plane, where $R_1 + R_2 + R_3 = \sqrt{2/3}$, and introduced the trilinear coordinates $x_j = R_j/(R_1 + R_2 + R_3)$. Here x_j denotes the side of a similar triangle with perimeter 1, i.e. $x_1 + x_2 + x_3 = 1$. The transformation from R_j to x_j is one-to-one, except in the parabolic case where there is a critical curve on the $\alpha\beta$ plane and every point on the critical curve corresponds to the radial line in the $\mathbf{R}$ space.

The $\alpha\beta$ coordinates in the plane are related to x_j by, $\beta = x_3$ and $\alpha = (x_2 - x_1)/\sqrt{3}$. The inverse transformation is unique, $x_3 = \beta$, $x_2 + x_1 = 1 - \beta$ and $x_1 = [1 - \beta - \alpha\sqrt{3}]/2$, $x_2 = [1 - \beta + \alpha\sqrt{3}]/2$. Owing to the triangle inequality, the admissible solutions x_j are confined in the $\alpha\beta$ plane to the triangle with vertices ($\pm 1/4$, $\sqrt{3}/4$) and (0,0) and its opposite side (back face) for vortex centers having the opposite orientation.

With the sum of the products of two different vortex strengths defined by, $K = k_1k_2 + k_2k_3 + k_3k_1$, the motions of the three vortices are classified [2] as elliptic, parabolic, or hyperbolic according as K is greater, equal, or less than zero, respectively.

The equilibrium points of the vortex configurations are either the equilateral triangle $E : x_j = 1/3$ or the collinear configurations. Their stabilities depend on the classification. For example, the equilateral configuration is stable (unstable) for the elliptic (hyperbolic) cases.

Synge presented two integral invariants $\sum_{j=1}^{3} k_j^{-1} R_j^2 = a$ and $\prod_{j+1}^{3} R_j^{1/k_j} = b$ for the motion in R_j space. This was reduced to a single integral invariant in the trilinear coordinates or for $\alpha\beta$ by Tavantzis and Ting [5] for $K \neq 0$ and one for $K = 0$. We shall replace the first one by its $-K/(2k_3)$- th power, so that the new invariant is valid for all K. It is

$$[\sum_{j=1}^{3}\frac{x_j^2}{k_j}]^{K/(2k_3)}[\prod_{j=1}^{3} x_j^{1/k_j}]^{k_1k_2} = const.\bar{I} \qquad (1)$$

When the original system is perturbed, the variation of $\bar{I}$ provides a measure of the deviation from the original problem.

We consider two types of perturbations: (i) perturbation of the initial data of the original problem, and (ii) three vortices in a half-plane considered as perturbation of the original problem.

A perturbation of type (i) leads to a small change of the constant $\bar{I}$, i.e. which then remains constant throughout the course of the motion. The initial trajectory moves to a neighboring one, accounting for a small change in $\bar{I}$. Therefore, we shall carry out only numerical studies of type (ii) in the next section.

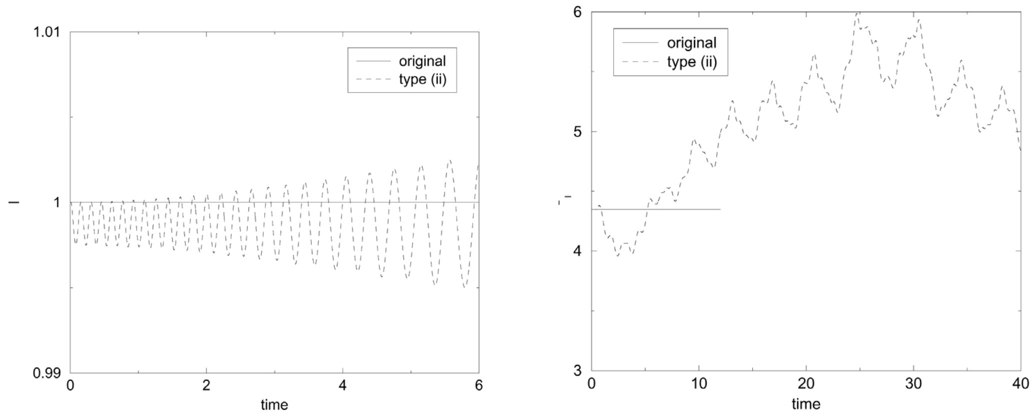

Fig. 1. Evolution of $\bar{I}$ for the parabolic (left) and hyperbolic (right) cases. Plotted are curves for the unperturbed system (solid) and for type (ii) perturbation (dash).

3. Numerical investigations

We study the perturbed problems for three vortices in a half-plane with the initial positions of the vortex centers forming an equilateral triangle. The initial positions are

$$z_1 = x_0 + 10.1667i, \quad z_2 = x_0 - 9.8333i, \quad z_3 = 10i,$$
$$\text{with} \quad x_0 = -1/(2\sqrt{3}) \tag{2}$$

Without loss of generality, we assume $k_2 = 1$ and $k_1 \geq k_2 \geq k_3$ and assign $k_1 = 2$ and $k_3 = -2/3$ for the parabolic case, where $K = 0$, and $k_1 = 2$, $k_3 = -1.25$ for the hyperbolic case with $K = -1.75$.

The variations of $\bar{I}$ from the invariant of the unperturbed problem for the parabolic and hyperbolic cases, respectively, are shown in Fig. 1. We see that although the perturbation due to the boundary of the half-plane is $O(1/30)$, the variations in $\bar{I}$ are finite and irregular, and more so for the hyperbolic case, for which the equilateral triangle corresponds to a (unstable) saddle point.

Some details of the deviations are shown in Figs. 2–4. Figure 2 and Fig. 4 show the motions of the vortices with respect to the center of vorticity (centroid-weighted by the vortex strengths) of the configuration in the complex z plane with x, y denoting the real and imaginary parts of z. For the unperturbed case, in Fig. 2 and in 4, the configuration remains almost periodic due to the high-order numerical accuracy. For the perturbed parabolic case, as shown in Fig. 2, the vortex configuration is expanding. Thus the perturbation moves the configuration to a stable expanding similar configuration. For the perturbed hyperbolic case, shown in Fig. 4, the motion is highly irregular. Figure 3 shows another aspect of the disturbed hyperbolic case via the variation of $\alpha(t)$, which is $(x_2 - x_1)\sqrt{3}$. For the undisturbed case, α looks periodic and nonpositive, that is, $R_2 \leq R_1$, while for the perturbed problem, we see that $\alpha(t)$ is aperiodic and changes sign, over a longer duration for $R_2 > R_1$.

Thus we have some numerical verification of the connection of the type of deviations of the perturbed problem, small and near periodic, or finite and aper-

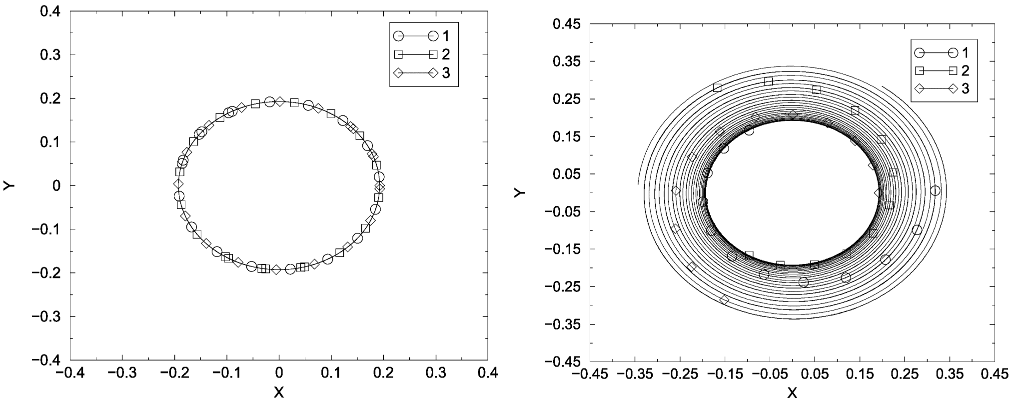

Fig. 2. Motion of the three vortices with respect to the centroid for the parabolic case, $k_3 = -2/3$. Left: original system; right: type (ii) perturbation.

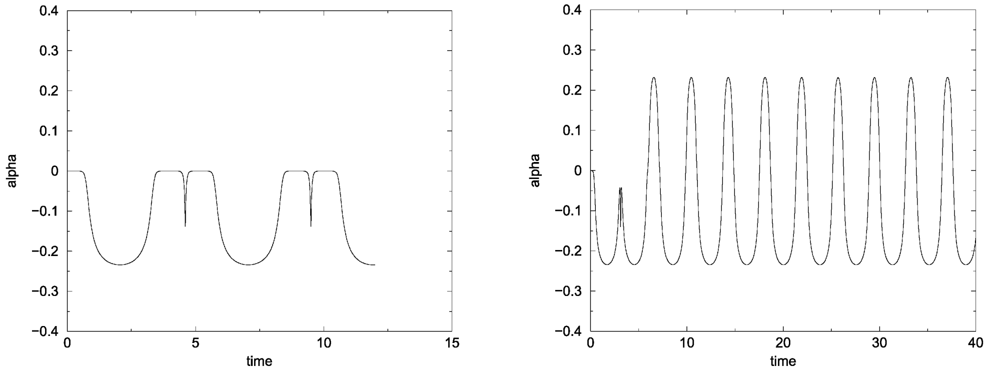

Fig. 3. Evolution of α for the hyperbolic case, $k_3 = -1.25$. Left: original system; right: type (ii) perturbation.

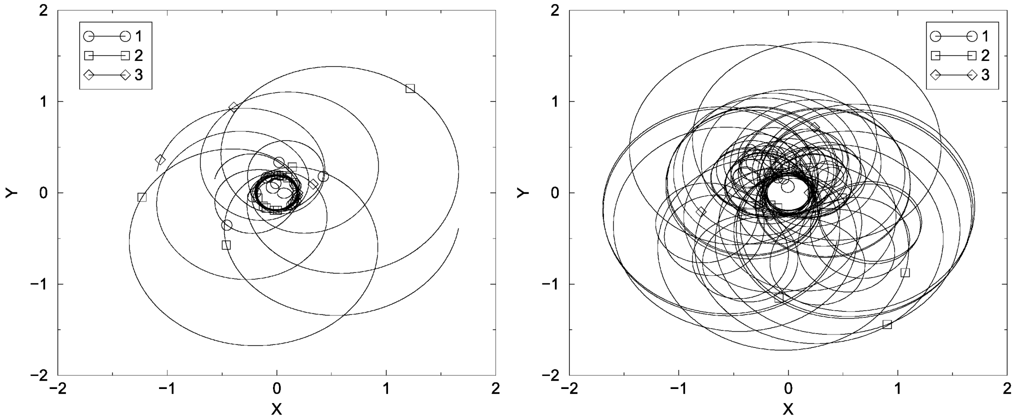

Fig. 4. Motion of the three vortices with respect to the centroid for the hyperbolic case, $k_3 = -1.25$. Left: original system; right: type (ii) perturbation.

iodic, to the stability of the original problem. Additional numerical studies of the perturbed problems near other critical points of the original problem [5] will be carried out in future investigations.

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