GLS-type finite element methods for viscoelastic fluid flow simulation

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Abstract

The stabilized Galerkin/Least-Squares finite element formulation for viscoelastic fluids circumvents inf-sup conditions on all three fields involved – stress, velocity and pressure – allowing the use of low- and equal-order interpolations, and provides necessary stability in the high Weissenberg number regime. A new definition of stabilization parameter for the stabilized form of the constitutive equation is evaluated using a benchmark problem of Oldroyd-B flow past a cylinder in a channel. To address the issue of weak consistency exhibited by low-order velocity interpolations in the context of stabilized formulations, we also employ velocity gradient recovery for the Newtonian solvent. We show that the proposed parameter improves the agreement of the GLS formulation results with standard DEVSS results, especially in the high-Weissenberg number limit. In contrast to DEVSS, fully-implicit velocity gradient computation is not crucial for stability.

Keywords: Stabilized FEM; Viscoelastic fluids; Galerkin/Least-Squares

1. Introduction

Viscoelastic fluids of rate type present a number of numerical challenges: the advective nature of the constitutive equations, and the interaction of multiple discrete unknown fields (viscoelastic stress, velocity and pressure). These obstacles are being gradually overcome: when using the finite element approach, adding the Streamline-Upwind/Petrov-Galerkin (SUPG) terms [1] to the Galerkin formulation was instrumental in overcoming the difficulties associated with the advective terms in the constitutive equation. The Discontinuous-Galerkin (DG) [2] approach provides similar benefits. Compatibility conditions on stress and velocity interpolations were formulated [3] and satisfied by complex combinations of interpolation functions, such as the 4 × 4 stress sub-element [1] complementing quadratic velocity interpolation. Alternative methods were soon developed which admitted simpler equal-order interpolations of the viscoelastic stress and velocity. Reviews by Baaijens [4] and Keunings [5] outline the development of the Elastic Viscous Stress Splitting (EVSS) family of methods. In particular, the Discrete EVSS approaches, such as DEVSS-G/SUPG [6,7] and DEVSS-G/DG [8], include the following features:

- the velocity gradient is approximated with continuous interpolation functions;
- the viscous stress is split into two contributions, associated with the continuous velocity gradient and with the discontinuous gradient of the velocity field;
- the constitutive equation of the viscoelastic stress (or the conformation) is discretized with a streamline-upwind (SUPG) or discontinuous Galerkin (DG) method.

One approach that has been quite successful in circumventing the compatibility condition in the case of the Navier-Stokes equations has been the Galerkin/Least-Squares (GLS) method [9], in which a stabilizing least-squares form of the governing equation is added to the Galerkin form. The resulting formulation recovers SUPG terms, and also alleviates the need to satisfy the compatibility condition. This particular approach has been used by Behr et al. [10] to design and analyze a Galerkin/Least-Squares variational formulation of Navier-Stokes equations involving viscous stress,

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velocity and pressure as the primary variables, without the usual restrictions on the interpolation function spaces. The GLS approach was extended by Behr [11] to the upper-convected Maxwell and Oldroyd-B constitutive models – hereafter referred to as three-field GLS formulation, or GLS3. The method showed a number of desirable characteristics:

- SUPG stabilization terms in the constitutive equation were obtained immediately from the GLS terms;
- equal-order interpolations for all flow field variables (viscoelastic stress, velocity and pressure) were admissible;
- hence, the implementation was straightforward and the computational cost was modest.

The formulation was augmented by several variants of continuous velocity gradient recovery to improve accuracy in [12]. Here, we continue the investigation of the GLS3 method from [12], addressing the discrepancies between GLS3 and DEVSS results at high Weissenberg number for a benchmark problem. An improved definition of the stabilization parameter is presented.

In the following, we review the equations governing the motion of an Oldroyd-B fluid in Section 2, present the three-field Galerkin/Least-Squares finite element formulation and its stabilization parameters in Section 3, and assess the performance of that formulation in Section 4, using an example of flow past a cylinder placed in a channel.

2. Governing equations

We consider an incompressible fluid occupying at an instant $t \in (0, T)$ a bounded region $\Omega_t \subset \mathbb{R}^{n_{sd}}$, with boundary Γ_t, where n_{sd} is the number of space dimensions. The velocity and pressure, $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$, are governed by the momentum and mass balance equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}\right) - \nabla\cdot\boldsymbol{\sigma} = \mathbf{0} \quad \text{on } \Omega_t \forall t \in (0, T) \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{on } \Omega_t \forall t \in (0, T) \tag{2}$$

where ρ is the fluid density, assumed to be constant, and $\mathbf{f}(\mathbf{x}, t)$ is an external force. The closure is obtained with a constitutive equation relating the stress tensor $\boldsymbol{\sigma}$ to velocity and pressure fields. Both the Dirichlet and Neumann-type boundary conditions are taken into account, represented as:

$$\mathbf{u} = \mathbf{g} \quad \text{on } (\Gamma_t)_g \tag{3}$$

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{h} \quad \text{on } (\Gamma_t)_h \tag{4}$$

where $(\Gamma_t)_g$ and $(\Gamma_t)_h$ are complementary subsets of the boundary Γ_t. The vector subscripts signify that this decomposition of Γ_t may be different for each of the velocity components. An appropriate initial condition $\mathbf{u}(\mathbf{x}, 0)$ is also specified.

Viscoelastic fluids exhibit dependence of the stress not only on the instantaneous rate of strain, but also on the strain history, as in the case of the upper-convected Oldroyd (Oldroyd-B) model:

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{T} \\ \mathbf{T} &= \mathbf{T}_1 + \mathbf{T}_2 \\ \mathbf{T}_1 + \lambda \overset{\nabla}{\mathbf{T}}_1 &= 2\mu_1 \boldsymbol{\varepsilon}(\mathbf{u}) \\ \mathbf{T}_2 &= 2\mu_2 \boldsymbol{\varepsilon}(\mathbf{u}) \end{aligned} \tag{5}$$

where the $\overset{\nabla}{\mathbf{T}}$ denotes an upper-convected derivative:

$$\overset{\nabla}{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{T} - \left(\nabla\mathbf{u}\cdot\mathbf{T} + \mathbf{T}\cdot(\nabla\mathbf{u})^T\right) \tag{6}$$

the rate-of-strain tensor is defined as:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{7}$$

and polymer viscosity μ_1 and solvent viscosity μ_2 are specified.

In the case of steady flows considered in the remainder of this article, the time derivatives in Eqs. (1) and (6) are dropped, and domain Ω_t is replaced by a constant region Ω.

3. Three-field GLS formulation

Components of the extra stress are treated as additional degrees of freedom, complementing the velocity and pressure fields in a mixed formulation. The choice of stress and velocity interpolation functions must satisfy certain compatibility conditions, as mentioned in Section 1. It is also well known that a Galerkin formulation of the constitutive equation remains convergent only in the small Weissenberg number regime.

The GLS3 velocity-pressure-stress formulation for Oldroyd-B fluid is written as follows: find $\mathbf{u}^h \in \mathcal{S}_u^h, p^h \in \mathcal{S}_p^h$ and $\mathbf{T}^h \in \mathcal{S}_{\mathrm{T}}^h$ such that:

$$\int_\Omega \mathbf{w}^h\cdot\rho\,(\mathbf{u}^h\cdot\nabla\mathbf{u}^h - \mathbf{f})\,d\Omega - \int_\Omega \nabla\cdot\mathbf{w}^h p^h d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \mathbf{T}^h d\Omega$$
$$+ \; 2\mu_2 \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\varepsilon}(\mathbf{u}^h) - \int_{\Gamma_h} \mathbf{w}^h\cdot\boldsymbol{h}^h d\Gamma d\Omega + \int_\Omega q^h \nabla\cdot\mathbf{u}^h d\Omega$$

$$+ \quad \frac{1}{2\mu_1}\int_\Omega \mathbf{S}^h : \mathbf{T}^h d\Omega + \frac{\lambda}{2\mu_1}\int_\Omega \mathbf{S}^h : \overset{\nabla}{\mathbf{T}}{}^h \; d\Omega - \int_\Omega \mathbf{S}^h : \boldsymbol{\varepsilon}(\mathbf{u}^h)\; d\Omega$$

$$+ \quad \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{MOM}} \frac{1}{\rho}[\rho\,(\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h) + \boldsymbol{\nabla} q^h - \boldsymbol{\nabla}\cdot\mathbf{S}^h - 2\mu_2 \boldsymbol{\nabla}\cdot\boldsymbol{\varepsilon}(\mathbf{w}^h)]$$

$$\cdot[\rho\,(\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h - \mathbf{f}) + \boldsymbol{\nabla} p^h - \boldsymbol{\nabla}\cdot\mathbf{T}^h - 2\mu_2\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}^h)]\; d\Omega$$

$$+ \quad \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{CONS}} 2\mu_1 \left[\frac{1}{2\mu_1}\mathbf{S}^h + \frac{\lambda}{2\mu_1}\overset{\nabla}{\mathbf{S}}{}^h - \boldsymbol{\varepsilon}(\mathbf{w}^h)\right]$$

$$: \left[\frac{1}{2\mu_1}\mathbf{T}^h + \frac{\lambda}{2\mu_1}\overset{\nabla}{\mathbf{T}}{}^h - \boldsymbol{\varepsilon}(\mathbf{u}^h)\right] d\Omega$$

$$+ \quad \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\text{CONT}} \boldsymbol{\nabla}\cdot\mathbf{w}^h \rho\, \boldsymbol{\nabla}\cdot\mathbf{u}^h d\Omega = 0, \quad \forall \mathbf{w}^h \in \mathcal{V}_u^h, \quad \forall q^h \in \mathcal{V}_p^h, \quad \forall \mathbf{S}^h \in \mathcal{V}_T^h \tag{8}$$

where $\mathcal{S}_u^h$, $\mathcal{S}_p^h$, $\mathcal{S}_T^h$ and $\mathcal{V}_u^h$, $\mathcal{V}_p^h$, $\mathcal{V}_T^h$, are appropriately defined interpolation and weighting function spaces for velocity, pressure, and extra stress [12].

The stabilization parameters τ_{MOM} and τ_{CONT} follow standard definitions given, e.g. in [13]. The parameter τ_{CONS} is taken here as:

$$\tau_{\text{CONS}} = \left(1 + \left(\frac{2\lambda|\mathbf{u}^h|}{h}\right)^2 + \left(\lambda|\boldsymbol{\nabla}\mathbf{u}^h|\right)^2\right)^{-\frac{1}{2}} \tag{9}$$

where h is the element length.

The least-squares form of the momentum equation, i.e. the τ_{MOM}-term in Eq. (8), counters the under-diffusivity of the Galerkin discretization at high Peclet numbers, and a possible lack of compatibility between velocity and pressure spaces. The least-squares form of the continuity equation, i.e. the τ_{CONT}-term in Eq (8), improves the convergence of non-linear solvers at high Reynolds numbers. Finally, the least-squares form of the constitutive equation, i.e. the τ_{CONS}-term in Eq. (8), counters the under-diffusivity of the Galerkin discretization at high Weissenberg numbers, and a possible lack of compatibility between velocity and stress spaces. The combination of the stabilization terms gives us freedom in selecting the interpolation spaces. In the example that follows, a piecewise bi-linear interpolation is used for all three fields.

In [12], two variations of GLS3 formulation were introduced, which involve decoupled recovery of continuous velocity gradient using consistent (GLS3-M) and lumped (GLS3-L) mass matrix. These improve consistency of the stabilization terms in the presence of Newtonian solvent (Oldroyd-B fluid).

4. Numerical example

We consider again the benchmark problem used in [8,12], i.e. flow of Oldroyd-B fluid past a circular cylinder placed between parallel fixed plates, with channel width being 8 times the cylinder diameter. Problem parameters, flow domain, and the computational mesh are identical to ones described in [12].

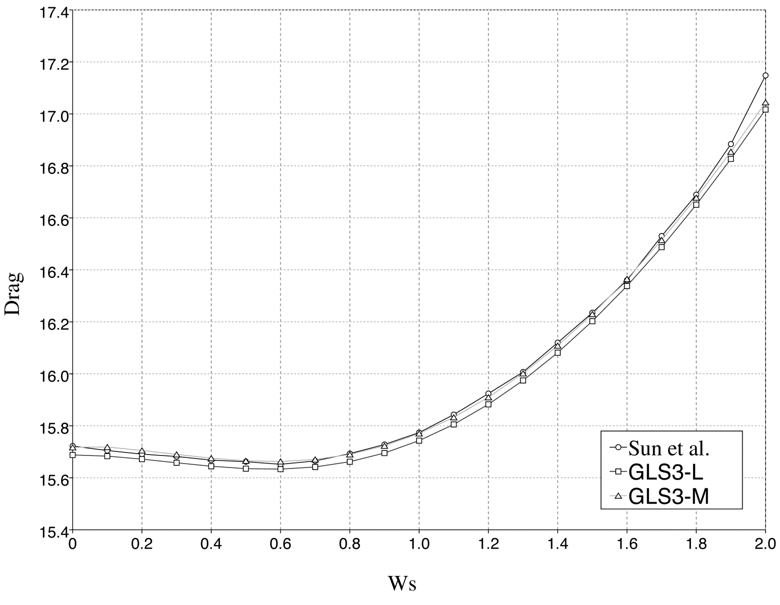

Fig. 1. Flow past a circular cylinder: drag as a function of Weissenberg number for GLS3 versus Sun et al. [8].

The drag for Weissenberg numbers 0.0 to 2.0 is shown in Fig. 1. The agreement between GLS3-M, GLS3-L and results of Sun et al. [8] is now excellent up to Weissenberg number of 1.8. The updated τ_{CONS} definition credited with eliminating discrepancies in drag computations in the high-Weissenberg number regime.

5. Summary

We have presented an evaluation of the stabilized three-field velocity-pressure-stress Galerkin/Least-Squares finite element formulation, using a new definition of the stabilization parameter for the constitutive equation. We have demonstrated – via comparison with an established DEVSS-G/DG approach – good agreement in measured drag force exerted on a cylinder by an Oldroyd-B fluid over a wide range of Weissenberg numbers.

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