Monte-Carlo simulations of baroclinic vortices

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Abstract

This paper examines the practicality of using a Monte-Carlo algorithm to analyse the point vortex equilibrium statistical model of two-layer baroclinic and barotropic quasi-geostrophic vortices in the two-dimensional plane. The conserved quantity, angular momentum, serves to confine the vortices.

Keywords: Monte-Carlo method; Equilibrium statistical mechanics; Heton model; Two-layer quasi-geostrophic flows; Angular momentum

1. Introduction

The Monte-Carlo algorithm had been previously used in the study of equilibrium statistical mechanics of vortices in fluid dynamics. In this paper, we extend its use to an unbounded two-layer baroclinic point vortex model. In this model, developed by DiBattista and Majda [1], the rotational invariance of the Hamiltonian and domain give rise to an angular momentum constraint. Using the point vortex Monte-Carlo method, we avoid the mean-field theory altogether. Alternatively, we can view the approach here as a fast and efficient numerical method for solving the mean field equations for this theory. The angular momentum constraint leads to a vortex distribution in a compact domain, which dispenses with complicated boundary conditions and far-field assumptions that are needed in more traditional ways of addressing the mean field equations.

2. Two-layer quasi-geostrophic vorticity equations

DiBattista and Madja [1] present a rather comprehensive discussion on the two species point vortex model of the two-layer quasi-geostrophic vorticity equations. In this model, the stratified fluid which is permitted to evolve in the unbounded plane is partitioned into two thin slabs, each of constant depth, density and temperature.

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The two layer quasi-geostrophic vorticity model give rise to the following conserved quantities:

$$Q_1 = \int_{R^2} q_1 \tag{1}$$

$$Q_2 = \int_{R^2} q_2 \tag{2}$$

$$H = \sum_{j=1}^{2} -\frac{1}{2}\int_{R^2} \psi_j q_j \tag{3}$$

$$\Gamma = \sum_{j=1}^{2} \int_{R^2} q_j |\mathbf{x}^2| \tag{4}$$

$q_1\,(\mathbf{x})$ and $q_2\,(\mathbf{x})$ are the vortex density of the upper and lower layers respectively. In this paper, we shall only consider positive vorticity, $q_i\,(\mathbf{x}) > 0$ for all $\mathbf{x}$, in view of the discussion in [1,2]. ψ_1 and ψ_2 are the stream functions on the upper and lower layers respectively. They are coupled through the relations:

$$q_1 = \Delta\psi_1 - (F\psi_1 - F\psi_2) \quad \text{and}$$
$$q_2 = \Delta\psi_2 + (F\psi_1 - F\psi_2)$$

where Δ denotes the horizontal Laplacian operator and F is related to the Rossby radius L_ρ by

$$F = \frac{1}{L_{\rho^2}}$$

The quantity H is the pseudo-energy of the vortex system. Γ is the angular momentum which is conserved as a consequence of the rotational invariance of the infinite plane.

1.1 Point vortex model

To allow numerical simulations of the two-layer quasi-geostrophic model, we shall discretise the vortex field. We represent q_1 and q_2 with N discrete point vortices in each layer. That is, there are $2N$ numbers of point vortices. N of them belongs to the upper layer, each with vorticity $\omega_1 = \frac{Q_1}{N}$; and the remaining N to the lower layer, each having vorticity $\omega_2 = \frac{Q_2}{N}$.

The discretised version of the pseudo-energy (3) would be:

$$\begin{aligned} H = & -\frac{\omega_A^2}{2}\sum_{i\neq j}\left[G_B(\mathbf{x}_i,\mathbf{x}_j)+G_T(\mathbf{x}_i,\mathbf{x}_j)\right] \\ & -\frac{\omega_B^2}{2}\sum_{m\neq n}\left[G_B(\mathbf{x}_m,\mathbf{x}_n)+G_T(\mathbf{x}_m,\mathbf{x}_n)\right] \\ & -\frac{\omega_A\omega_B}{2}\sum_{i\neq m}\left[G_B(\mathbf{x}_i,\mathbf{x}_m)-G_T(\mathbf{x}_i,\mathbf{x}_m)\right] \end{aligned} \quad (5)$$

and that for the angular momentum (4):

$$\Gamma = \omega_A \sum_i \mathbf{x}_i^2 + \omega_B \sum_m \mathbf{x}_m^2. \quad (6)$$

In the above, the subscripts i and j denote vortices of upper layer while m and n denote the lower layer. G_B and G_T are the Green's functions for the barotropic $(\frac{q_1+q_2}{2})$ and baroclinic $(\frac{q_1-q_2}{2})$ vorticity field respectively. They are given by [1]:

$$\begin{aligned} G_B(\mathbf{x}_i,\mathbf{x}_j) &= \frac{1}{2\pi}\ln\left|\mathbf{x}_i,\mathbf{x}_j\right| \\ G_T(\mathbf{x}_i,\mathbf{x}_j) &= -\frac{1}{2\pi}K_0\left(\sqrt{2F}\left|\mathbf{x}_i,\mathbf{x}_j\right|\right) \end{aligned}$$

K_0 is the zeroth order modified Bessel function of the second kind [5], which effectively vanish for large distances.

1.2 Equilibrium statistical mechanics

From the point of view of equilibrium statistical mechanics, the point vortices can take any configuration they want, but each with a certain probability. The probability of a $2N$ vortex configuration $\vec{z}_N$ is given by:

$$P(\vec{z}_N) = \frac{\exp\{-\beta H(\vec{z}_N) - \mu\Gamma(\vec{z}_N)\}}{\int \exp\{-\beta H(\vec{z}_N) - \mu\Gamma(\vec{z}_N)\}\, d\vec{z}_N} \quad (7)$$

β and μ are the Lagrange multipliers associated with the conservation of H and Γ respectively.

2. Monte-Carlo algorithm

In all the experiments the number of vortices of species A is set equal to the number of vortices of species B. We take $F = 1$. We describe the results of one run here with $\beta = 1$ and $\mu = 1$. $\lambda_A = 1$ and $\lambda_B = 4$. Profile B is nearly a radial flat-top distribution, while profile A is nearer to a Gaussian.

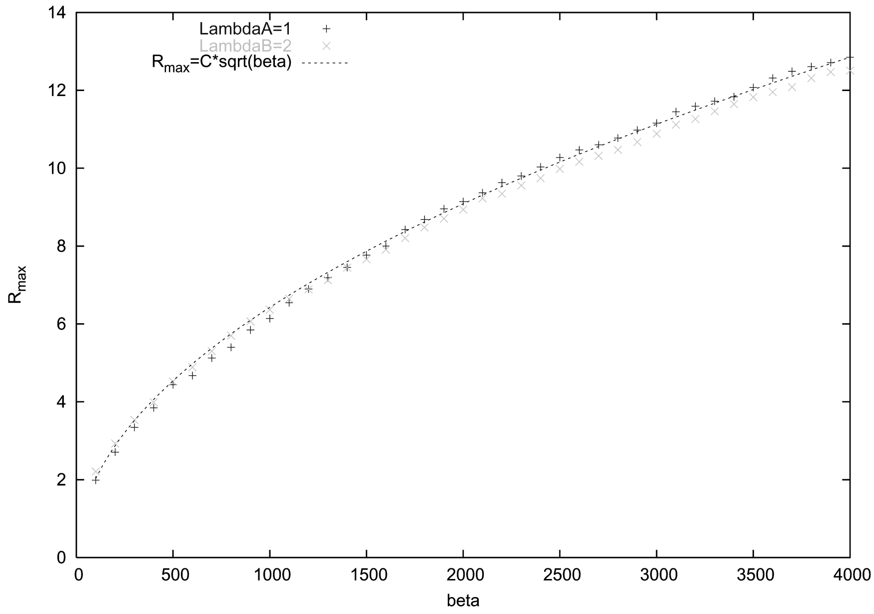

Fig. 1. Graph of R_{Max} vs β (Baroclinic). $C = 0.2032$.

Compiling several runs with fixed N and vortex strengths but with varying β over a wide range of positive values, we obtained the power law in Fig. 1 between the radius of the furthest vortex and β.

3. Conclusion

For $\beta = 0$, the point vortex Monte-Carlo algorithm produces results that are very close to the known exact results. There is a continuous variation in the radial vortex distribution when β is non-zero.

We conclude with a few specific physical points which are worth following up in future works. They are (1) as the vortex strength of species B gets progressively larger with respect to species A, the equilibrium distributions of species B (lower layer) become closer to a flat-top radial distribution while that of species A remains nearer to a Gaussian distribution. (2) The simultaneous appearance of both canonical types of vorticity distributions at a single positive temperature is remarkable. We refer the reader to recent papers [3,4] which discuss these canonical types in detail.

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