Time-harmonic analysis of a planar crack in an elastic half-space by BEM

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Abstract

The symmetric crack problem in an elastic half-space subjected to a time-harmonic crack-surface loading is investigated. For this purpose, a boundary element method (BEM) is developed, which contains only a boundary integral over the crack surface. The traction-free conditions on the half-space boundary are satisfied identically in the method. Numerical results for the mode-I dynamic stress intensity factor of a penny-shaped crack are presented to analyze the effects of the frequency, the crack location, and the reflected waves by the half-space boundary.

Keywords: Time-harmonic crack analysis; Elastic half-space; Penny-shaped crack; Dynamic stress intensity factor; Boundary element method

1. Introduction

Elastic wave propagation in cracked solids is of particular interest to fracture mechanics and ultrasonic non-destructive material testing [1,2]. Among the many numerical methods the boundary integral equation method (BIEM) or the boundary element method (BEM) provides an accurate and efficient numerical tool for wave propagation simulation in cracked elastic solids. In this paper a frequency-domain BEM is proposed and applied to a symmetric crack problem in a three-dimensional elastic half-space. The method contains only a boundary integral over the crack surface, while the traction-free conditions on the half-space boundary are satisfied automatically. A brief discussion on the computation of hypersingular and weakly singular integrals is given. Numerical results for the mode-I dynamic stress intensity factor are presented. Special attention of the analysis is devoted to the investigation of the effects of the frequency, the crack location and the reflected elastic waves on the mode-I dynamic stress intensity factor.

2. Problem statement and boundary integral equation

Let us consider a homogeneous, isotropic and linear elastic half-space $x_2 \leq 0$ containing a crack S_c located in the plane $x_3 = 0$ and perpendicular to the boundary of the half-space S as depicted in Fig. 1. The crack is subjected to a tensile time-harmonic crack-surface loading $\bar{\sigma}_{33}(\mathbf{x},t) = N(\mathbf{x})\exp(-i\omega t)$, where $N(\mathbf{x})$ is the loading amplitude, ω is the circular frequency, and t is the time. Throughout the analysis, the common factor $\exp(-i\omega t)$ is suppressed. The half-space surface S is traction-free.

The total displacement wave field in such a solid can be written as:

$$u_i^{tot}(\mathbf{x}) = u_i(\mathbf{x}) + u_i^0(\mathbf{x}) \tag{1}$$

where $u_i(\mathbf{x})$ denotes the primary wave field caused by the crack-surface loading and $u_i^0(\mathbf{x})$ represents the reflected wave field by the half-space boundary. The boundary conditions on S and S_c can be stated as:

$$\sigma_{j2}(\mathbf{x}) = -\sigma_{j2}^0(\mathbf{x}), \quad \text{for} \quad \mathrm{x} \in S \tag{2}$$

$$\sigma_{33}(\mathbf{x}) = -N(\mathbf{x}) - \sigma_{33}^0(\mathbf{x}), \quad \sigma_{\alpha 3}(\mathbf{x}) = -\sigma_{\alpha 3}^0(\mathbf{x}) \quad \text{for} \quad \mathrm{x} \in S_c \tag{3}$$

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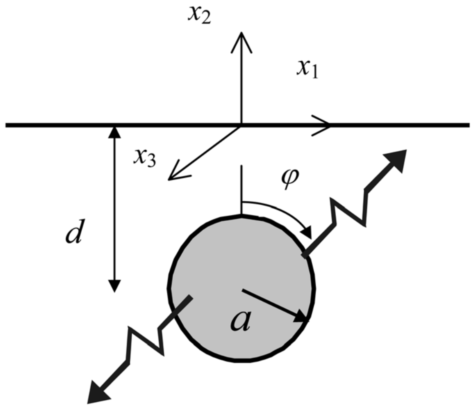

Fig. 1. A planar crack in an elastic half-space subjected to a crack-surface loading.

The boundary value problem, as stated above, can be solved by using a boundary integral equation formulation. For this purpose the following integral representations for the displacement components are used:

$$u_j(\mathbf{x}) = \frac{\partial H_1}{\partial x_j} + 2(\delta_{j3}-1)\frac{\partial H_2}{\partial x_j} + \frac{2}{k_2^2}\frac{\partial}{\partial x_j}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)(H_1 - H_2)$$

$$u_j^0(\mathbf{x}) = \frac{\partial H_{j2}}{\partial x_2} + \frac{\partial (H_{21}-2H_{22})}{\partial x_j} + \delta_{j2}\left(\frac{\partial H_{12}}{\partial x_1}+\frac{\partial H_{22}}{\partial x_2}+\frac{\partial H_{32}}{\partial x_3}\right) + \frac{2}{k_2^2}\frac{\partial}{\partial x_j}\left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}\right)(H_{21}-H_{22}) - \frac{\partial}{\partial x_2}\left(\frac{\partial (H_{11}-H_{12})}{\partial x_1}+\frac{\partial (H_{31}-H_{32})}{\partial x_3}\right)\right] \quad (4)$$

where H_k and H_{kj} are the Helmholtz's potentials

$$H_\alpha(\mathbf{x}) = \iint_{S_c} \Delta u_3(\mathbf{y})\frac{\exp(ik_\alpha r)}{r}dS_{\mathbf{y}}$$

$$H_{j\alpha}(\mathbf{x}) = \iint_{S} f_j(\mathbf{y})\frac{\exp(ik_\alpha r)}{r}dS_{\mathbf{y}} \quad (5)$$

In Eq. (5), $r = |\mathbf{x}-\mathbf{y}|$, $k_\alpha = \omega/c_\alpha$, are the wave numbers, c_1 and c_2 are the longitudinal and the transverse wave velocities, Δu_3 is the unknown normal crack-opening-displacement, and f_j are the unknown boundary densities on S. Note here that the tangential crack-opening displacements vanish because of the symmetry of the problem.

By substituting Eq. (4) into Hooke's law and invoking the boundary conditions Eq. (2), the potentials of reflected waves H_{jk} can be established analytically. A subsequent use of Fourier transform technique with respect to the variables x_1 and x_2 in Eq. (2) yields, finally, the following integral relations between the boundary densities f_j and the crack-opening-displacement Δu_3 (implicitly contained in the stress components σ_{j2}):

$$f_j(\mathbf{x}) = \frac{1}{4G\pi^2}\sum_{n=1}^{3}\iint_S \sigma_{n2}(\eta)\int_0^\infty \frac{\tau}{R(\tau)}\mathrm{U}_{jn}^{\eta,\tau}[J_0(\tau|\mathbf{x}-\eta|)]d\tau dS_\eta, \quad \mathbf{x}\in\mathcal{S} \quad (6)$$

where G is the shear modulus, $J_0(\cdot)$ is the Bessel function of the first kind and zeroth order, and $\mathrm{R}(\tau) = (\tau^2 - \mathrm{k}_2^2/2)^2 - \tau^2\sqrt{\tau^2-\mathrm{k}_1^2}\sqrt{\tau^2-\mathrm{k}_2^2}$ is the Rayleigh function. The expressions for the differential operators $\mathrm{U}_{jn}^{\eta,\tau}$ are given by Mykhas'kiv et al. [3].

After substituting Eqs. (4)–(6) into Hooke's law and then invoking the boundary conditions in Eq. (3), the following BIE is obtained for Δu_3:

$$\iint_{S_c}[L_{\mathrm{inf}}(r) - L_{\mathrm{int}}(\mathbf{x},\mathbf{y})]\Delta u_3(\mathbf{y})dS_{\mathbf{y}} = \frac{k_2^2}{4G}N(\mathbf{x}), \quad \mathbf{x}\in S_c \quad (7)$$

Note here that the BIE (7) contains only an integral over the crack surface, which is efficient for the numerical solution. The Helmholtz's potential singularities are contained in the kernel L_{inf}, which is the same as for a crack in an infinite solid:

$$L_{\mathrm{inf}}(r) = \left[9 - 9ik_1r + (k_2^2-5k_1^2)r^2 + ik_1(2k_1^2-k_2^2)r^3 + \frac{1}{4}(2k_1^2-k_2^2)^2r^4\right]\frac{\exp(ik_1r)}{r^5} - \left[9 - 9ik_2r - 4k_2^2r^2 + ik_2^3r^3\right]\frac{\exp(ik_2r)}{r^5} \quad (8)$$

The kernel L_{int} considering the interaction between the crack and the half-space boundary is given by:

$$L_{\mathrm{int}}(\mathbf{x},\mathbf{y}) = L_{\mathrm{inf}}(|\bar{\mathbf{x}}-\mathbf{y}|) + 2\int_0^\infty \frac{\tau}{R(\tau)}\Omega(\mathbf{x},\mathbf{y},\tau)d\tau, \quad \bar{\mathbf{x}} = (x_1, -x_2) \quad (9)$$

where Ω is a known regular function (see [3]).

3. Numerical solution of the boundary integral equation

To isolate the singularities explicitly in the kernel L_{inf}, the singularity subtraction technique is applied to Eq. (7) by using static potentials with a hypersingularity and a weak singularity. This results in:

$$\iint_{S_c} \frac{\Delta u_3(\mathbf{y})}{r^3} dS_{\mathbf{y}} + Ak_2^2 \iint_{S_c} \frac{\Delta u_3(\mathbf{y})}{r} dS_{\mathbf{y}} + \iint_{S_c} \left[\frac{4(1-\nu)}{k_2^2} L_{\text{inf}}(r) - \frac{1}{r^3} - \frac{Ak_2^2}{r}\right] \Delta u_3(\mathbf{y}) dS_{\mathbf{y}} - \frac{4(1-\nu)}{k_2^2} \iint_{S_c} L_{\text{int}}(\mathbf{x},\mathbf{y})\, \Delta u_3(\mathbf{y}) dS_{\mathbf{y}} = \frac{1-\nu}{G} N(\mathbf{x}),$$

$$\mathbf{x} \in S_c, \quad A = \frac{7 - 12\nu + 8\nu^2}{8(1-\nu)} \tag{10}$$

where ν is Poisson's ratio. To describe the local behavior of the crack-opening displacement at the crack-front correctly, the following ansatz is used in the case of a penny-shaped crack with the radius a and the center at $O(0,-d)$.

$$\Delta u_3(\mathbf{x}) = \sqrt{a^2 - x_1^2 - (x_2 + d)^2}\, \alpha(\mathbf{x}), \qquad \mathbf{x} \in S_c \tag{11}$$

where $\alpha(\mathbf{x})$ is a new unknown smooth function. The hypersingular and weakly singular integrals of Eq. (10) can be regularized by using the following identities:

$$\iint_{S_c} \frac{\sqrt{a^2 - y_1^2 - (y_2 + d)^2}}{|\mathbf{x} - \mathbf{y}|^3} \alpha(\mathbf{y})\, dS_{\mathbf{y}} = -\pi^2 \alpha(\mathbf{x}) - \frac{1}{2}\pi^2 x_1 \frac{\partial \alpha(\mathbf{x})}{\partial x_1} - \frac{1}{2}\pi^2 (x_2 + d) \frac{\partial \alpha(\mathbf{x})}{\partial x_2} + \frac{\pi^2}{32}\left[4a^2 - x_1^2 - 3(x_2 + d)^2\right] \frac{\partial^2 \alpha(\mathbf{x})}{\partial x_1^2} + \frac{\pi^2}{8} x_1 (x_2 + d) \frac{\partial^2 \alpha(\mathbf{x})}{\partial x_1 \partial x_2} + \frac{\pi^2}{32}\left[4a^2 - 3x_1^2 - (x_2 + d)^2\right] \times \frac{\partial^2 \alpha(\mathbf{x})}{\partial x_2^2} + \iint_{S_c} \frac{\sqrt{a^2 - y_1^2 - (y_2 + d)^2}}{|\mathbf{x} - \mathbf{y}|^3} [\alpha(\mathbf{y}) - \alpha(\mathbf{x}) - (y_1 - x_1)\frac{\partial \alpha(\mathbf{x})}{\partial x_1} - (y_2 - x_2)\frac{\partial \alpha(\mathbf{x})}{\partial x_2} - \frac{1}{2}(y_1 - x_1)^2 \frac{\partial^2 \alpha(\mathbf{x})}{\partial x_1^2} - (y_1 - x_1)(y_2 - x_2)\frac{\partial^2 \alpha(\mathbf{x})}{\partial x_1 \partial x_2} - \frac{1}{2}(y_2 - x_2)^2 \frac{\partial^2 \alpha(\mathbf{x})}{\partial x_2^2}]\, dS_{\mathbf{y}}$$

$$\iint_{S_c} \frac{\sqrt{a^2 - y_1^2 - (y_2 + d)^2}}{|\mathbf{x} - \mathbf{y}|} \alpha(\mathbf{y})\, dS_{\mathbf{y}} = \frac{\pi^2}{4}\left[2a^2 - x_1^2 - (x_2 + d)^2\right] \alpha(\mathbf{x}) + \iint_{S_c} \frac{\sqrt{a^2 - y_1^2 - (y_2 + d)^2}}{|\mathbf{x} - \mathbf{y}|} [\alpha(\mathbf{y}) - \alpha(\mathbf{x})]\, dS_{\mathbf{y}} \tag{12}$$

For the evaluation of the kernel L_{int} by Eq. (9), the pole of the integrand due to the existence of the real root of the function R at $\tau = \omega / c_R$ should be considered, where c_R is the Rayleigh wave velocity ($c_R < c_2 < c_1$). To calculate the integral with this pole the following identical transform is applied:

$$\int_{k_2}^{\infty} \frac{\tau}{R(\tau)} \Omega(\mathbf{x},\mathbf{y},\tau) d\tau = -\frac{2(1-\nu)}{k_2^6} \int_1^{\infty} \frac{\eta W(\mathbf{x},\mathbf{y},\eta)}{(\eta^2 - \xi_R^2) V(\eta)} d\eta = -\frac{2(1-\nu)}{k_2^6} \left\langle \frac{W(\mathbf{x},\mathbf{y},\xi_R)}{4B} \times \left\{ \ln \frac{V(1)}{(1 - \xi_R^2)^2} - \frac{2C}{\sqrt{D}} \left[\frac{\pi^2}{2} - \arctan \frac{-(\nu + \xi_R^2)}{\sqrt{D}}\right] \right\} + \int_1^{\infty} \frac{\eta [W(\mathbf{x},\mathbf{y},\eta) - W(\mathbf{x},\mathbf{y},\xi_R)]}{(\eta^2 - \xi_R^2) V(\eta)} d\eta \right\rangle \tag{13}$$

Here:

$$W(\mathbf{x},\mathbf{y},\eta) = \left[(\tau^2 - k_2^2/2)^2 + \tau^2 \sqrt{\tau^2 - k_l^2} \sqrt{\tau^2 - k_2^2}\right] \times \Omega(\mathbf{x},\mathbf{y},\eta), \quad \xi_R = c_2 / c_R > 1$$

$$V(\eta) = \eta^4 - (2 - \nu - \xi_R^2)\eta^2 - (2 - \nu)\xi_R^2 + \xi_R^4 + 1 - \nu$$

$$B = 2(2 - \nu)\xi_R^2 - 3\xi_R^4 - 1 + \nu, \quad C = 2 - \nu - 3\xi_R^2,$$

$$D = 3\xi_R^4 - 2(2 - \nu)\xi_R^2 - \nu^2 \tag{14}$$

By collocating Eq. (10) at discrete points, a system of linear algebraic equations is obtained. For the discretization of the crack surface, a boundary element mesh with a uniform division in the polar coordinate direction is used.

The mode-I dynamic stress intensity factor $K_I(\varphi,t)$ is determined by using the following relation:

$$K_I(\varphi,t) = -\frac{2G\pi\sqrt{\pi a}}{1-\nu} \alpha(\mathbf{x})\big|_{\substack{x_1 = a\sin\varphi \\ x_2 = -d + a\cos\varphi}} \exp(-i\omega t) \tag{15}$$

4. Numerical results

Numerical calculations have been carried out for a penny-shaped crack subjected to a uniform crack-surface loading $N(\mathbf{x}) = N_0 = \text{const}$. One hundred and sixty-one constant elements have been used, and

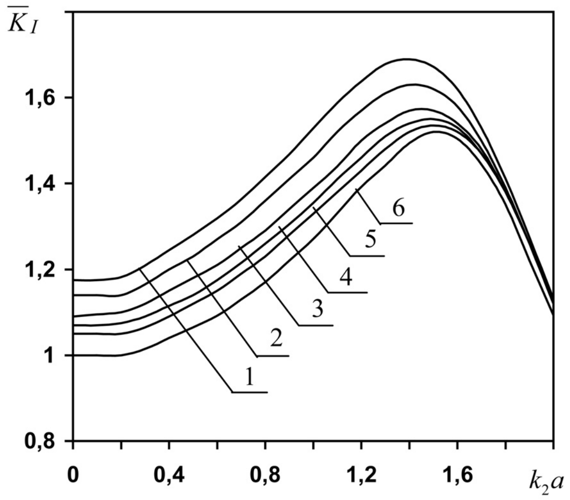

Fig. 2. $\bar{K}_I$-factor versus dimensionless wave number for a penny-shaped crack at $\varphi = 0°$. (1 – $d/a = 1.15$; 2 – $d/a = 1.2$; 3 – $d/a = 1.3$; 4 – $d/a = 1.4$; 5 – $d/a = 1.5$; 6 – infinite cracked solid)

Poisson's ratio has been taken as $\nu = 0.3$. For convenience, the normalized amplitude of the mode-I dynamic stress intensity factor $\bar{K}_I = |K_I|/K_I^{st}$ is introduced, where $K_I^{st} = 2N_0\sqrt{a/\pi}$. Figure 2 shows that at the crack-front nearest to the half-space boundary, $\bar{K}_I$ exceeds the corresponding value for a crack in an infinite solid. This conclusion is also valid for all points at the crack front in the low-frequency regime as can be seen in Fig. 3(a). At high frequencies, however, the existence of the half-space boundary may increase or decrease the mode-I dynamic stress intensity factor $\bar{K}_I$, depending on the considered position at the crack-front, see Fig. 3(b).

5. Conclusions

An improved BIE method for time-harmonic crack analysis in an elastic half-space is presented. The method requires only the discretization of the crack-surface and is efficient for treating symmetric crack problems in an elastic half-space. Numerical results are presented to show the effects of the frequency, the crack location, and the reflected waves by the free boundary of the half-space on the stress intensity factor.

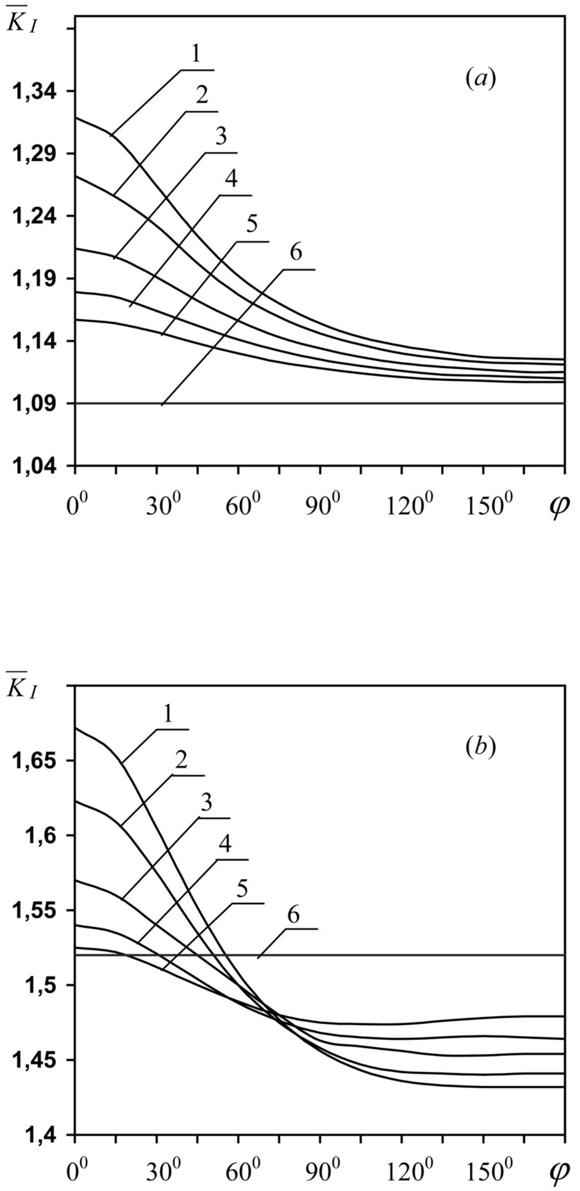

Fig. 3 $\bar{K}_I$-factor versus the polar angle along the crack-front, (a): $k_2a = 0.6$; (b) $k_2a = 1.5$. (1 – $d/a = 1.15$; 2 – $d/a = 1.2$; 3 – $d/a = 1.3$; 4 – $d/a = 1.4$; 5 – $d/a = 1.5$; 6 – infinite cracked solid)

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