Dimension reduction at most probable point for higher-order reliability analysis

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Abstract

A new dimension-reduction method based on most probable point (MPP) is presented for predicting reliability of mechanical systems subject to random loads, material properties, and geometry. The method involves univariate function representation at MPP, approximate response surface generation using dimension reduction, and Monte-Carlo simulation. The method yields higher-order approximation of a performance function without requiring any gradients. Results of two numerical examples involving an elementary mathematical function and a structural truss problem indicate that the proposed method provides accurate and computationally efficient estimates of the probability of failure.

Keywords: Reliability; Most probable point; FORM/SORM; Monte-Carlo simulation

1. Introduction

A fundamental problem in reliability analysis entails calculation of a multi-fold integral [1]

$$P_F \equiv \Pr[g(\boldsymbol{X}) < 0] = \int_{g(x)<0} f_x(\boldsymbol{x}) d\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{X} = \{X_1, \cdots, X_N\}^T \in \Re^N$ is an input random vector representing loads, material properties, and geometry, with joint probability density function, $f_x(\boldsymbol{x})$, $g(\boldsymbol{x})$ is the performance function, such that $g(\boldsymbol{x}) < 0$ represents the failure domain, and P_F is the probability of failure. The most common approach to compute the failure probability in Eq. 1 involves the first- and second-order reliability methods (FORM/SORM), which are based on linear (FORM) or quadratic approximation (SORM) of the limit-state surface at a most probable point (MPP). Experience has shown that FORM/SORM are sufficiently accurate for engineering purposes, provided that the limit-state surface at MPP is close to being linear or quadratic, and no multiple MPPs exist [1]. Otherwise, the results of FORM/SORM should be interpreted with caution. Recently, the authors have developed new dimension-reduction methods, which can solve highly nonlinear reliability problems more accurately or more efficiently than FORM/SORM and other simulation methods [2]. A major advantage of the dimension-reduction methods, so far based on mean point of random input, over FORM/SORM is that higher-order approximations of performance functions can be achieved without calculating MPP or gradients. However, for a certain class of reliability problems, existing dimension-reduction methods may require computationally demanding higher-variate (bivariate, trivariate, etc.) reductions to adequately represent performance functions. Hence, developing univariate dimension-reduction methods, which are capable of producing computationally efficient, yet sufficiently adequate performance functions, is the major motivation of the current work.

This paper presents an MPP-based univariate dimension-reduction method for predicting reliability of mechanical systems subject to random loads, material properties, and geometry. The method involves univariate function representation at MPP, approximate response surface generation using dimension reduction, and Monte-Carlo simulation. Numerical examples are presented to illustrate the proposed method.

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2. Dimension-reduction at most probable point

2.1. Univariate function representation

Consider a continuous, differentiable, real-valued performance function $g(\boldsymbol{x})$ that depends on $\boldsymbol{x} = \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N\} \in \Re^N$. The transformed limit states $h(\boldsymbol{u}) = 0$ and $y(\boldsymbol{v}) = 0$ are the maps of the original limit state $g(\boldsymbol{x}) = 0$ in the standard Gaussian space ($\boldsymbol{u}$ space) and the rotated Gaussian space (ν space), respectively, as shown in Fig. 1 for $N = 2$. The closest point of the limit-state surface to the origin, denoted the MPP ($\boldsymbol{u}^*$ or $\boldsymbol{v}^*$) or beta point, has a distance β, commonly referred to as the reliability index. The determination of MPP and β involves standard nonlinear constrained optimization and is usually performed in the standard Gaussian space [1]. Fig. 1 depicts FORM and SORM approximations of the limit-state surface at MPP.

Suppose that the performance function $y(\boldsymbol{v}) = 0$ has a convergent Taylor expansion at MPP $\boldsymbol{v}^* = \{v_1{}^*,\cdots,v_N{}^*\}^{\mathrm{T}}$, which can be expressed by:

$$y(\boldsymbol{v}) = y(\boldsymbol{v}^*) + \sum_{j=1}^{\infty}\frac{1}{j!}\sum_{i=1}^{N}\frac{\partial^j y}{\partial v_i^j}(\boldsymbol{v}^*)(v_i - v_i{}^*)^j + R_2 \tag{2}$$

where the remainder R_2 denotes all terms with dimension two and higher. Consider a univariate approximation of $y(\boldsymbol{v})$, denoted by:

$$\hat{y}_1(v) \equiv \hat{y}_1(v_1,\cdots,v_N) = \sum_{i=1}^{N} y(v_1{}^*,\cdots,v_{i-1}{}^*,v_i,v_{i+1}{}^*,\cdots,v_N{}^*) - (N-1)y(\boldsymbol{v}^*) \tag{3}$$

where each term in the summation is a function of only one variable and can be subsequently expanded in a Taylor series at $\boldsymbol{v} = \boldsymbol{v}^*$, yielding:

$$\hat{y}_1(\boldsymbol{v}) = y(\boldsymbol{v}^*) + \sum_{j=1}^{\infty}\frac{1}{j!}\sum_{i=1}^{N}\frac{\partial^j y}{\partial x_i^j}(\boldsymbol{v}^*)(v_i - v_i{}^*)^j \tag{4}$$

Comparing Eqs. 2 and 4 indicates that the univariate approximation leads to the residual error $y(\boldsymbol{v}) - \hat{y}_1(\boldsymbol{v}) = R_2$, which includes contributions from terms of dimension two and higher. For sufficiently smooth $y(\boldsymbol{v})$ with convergent Taylor series, the coefficients associated with higher-dimensional terms are usually much smaller than that with one-dimensional terms. In that case, higher-dimensional terms contribute less to the function and, therefore, can be neglected. Nevertheless, Eq. 3 includes all higher-order univariate terms, as compared with FORM and SORM, which only retain linear and quadratic terms, respectively. Hence, $\hat{y}_1(\boldsymbol{v})$ yields more accurate representation of $y(\boldsymbol{v})$ than FORM/SORM. Furthermore, Eq. 3 represents exactly the same function as $y(\boldsymbol{v})$ when $y(\boldsymbol{v}) = \sum_{i=1}^{N} y_i(v_i)$, when $y(\boldsymbol{v})$ can be additively decomposed into functions $y_i(v_i)$ of single variables.

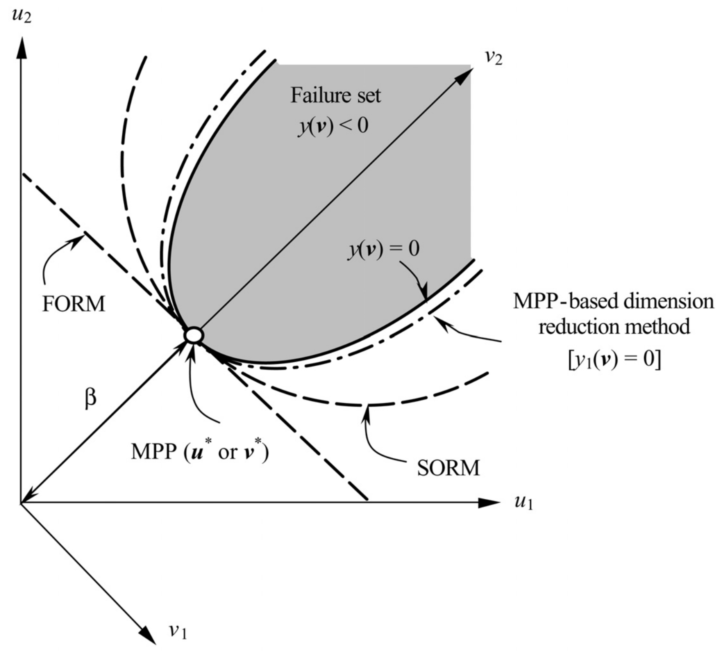

Fig. 1. Performance function approximations at MPP by various methods.

2.2. Response surface approximation

Consider the univariate terms $y_i(v_i) \equiv y(v_1{}^*,\ldots,v_{i-1}{}^*,v_i,v_{i+1}{}^*,\ldots,v_N{}^*)$ in Eq. 3. If for $v_i = v_i^{(j)}$, n function values:

$$y_i(v_i^{(j)}) = y(v_1{}^*,\cdots,v_{i-1}{}^*,v_i^{(j)},v_{i+1}{}^*,\cdots,v_N{}^*); j = 1,2,\cdots,n \tag{5}$$

are given, the function value for arbitrary v_i can be obtained using the Lagrange interpolation as:

$$y_i(v_i) = \sum_{j=1}^{n} \Phi_j(v_i) y_i(v_i^{(j)}) \tag{6}$$

where the shape function $\phi_j(\nu_i)$ is defined as:

$$\Phi_j(v_i) = \frac{\prod\limits_{k=1,k\neq j}^{n} \left(v_i - v_i^{(k)}\right)}{\prod\limits_{k=1,k\neq j}^{n} \left(v_i^{(j)} - v_i^{(k)}\right)} \tag{7}$$

By using Eq. 6, arbitrarily many values of $y_i(v_i)$ can be generated if n function values are given. Therefore, the total cost for univariate approximation entails $nN + 1$ function evaluations. More accurate multivariate approximations can be developed in the similar way. However, because of much higher cost, only univariate approximation will be examined in this paper.

2.3. Monte-Carlo simulation

For component reliability analysis, the Monte-Carlo estimate $P_{F,1}$ of the failure probability employing the proposed univariate representation is:

$$P_{F,1} = \frac{1}{N_S} \sum_{i=1}^{N_S} I\left[\hat{y}_1\left(\boldsymbol{v}^{(i)}\right) < 0\right] \tag{8}$$

where $\boldsymbol{v}^{(i)}$ is the ith realization of $\boldsymbol{V}$, N_S is the sample size, and $I[\cdot]$ is an indicator function such that $I = 1$ if $\boldsymbol{v}^{(i)}$ is in the failure set (i.e. when $\hat{y}_1(\boldsymbol{v}^{(i)}) < 0$) and *zero* otherwise.

3. Numerical examples

3.1. Example 1 – mathematical function

Consider a quartic performance function:

$$g(X_1, X_2) = \frac{5}{2} + \frac{1}{216}(X_1 + X_2 - 20)^4 - \frac{33}{140}(X_1 - X_2) \tag{9}$$

where $X_i \mapsto N(10, 3), i = 1, 2$ are independent, Gaussian random variables, each with mean $\mu = 10$ and standard deviation $\sigma = 3$. From the MPP search involving finite-difference gradients, the MPP in the rotated Gaussian space is $\boldsymbol{v}^* = \{0,2.5\}^T$ and $\beta = \|\boldsymbol{v}^*\| = 2.5$, as shown in Fig. 2. In addition, Fig. 2 plots limit states represented by various approximate reliability methods considered in this paper. For the dimension-reduction methods, five uniformly distributed points between $\mu - 2\sigma$ and $\mu + 2\sigma$ were deployed at each dimension (i.e. $n = 5$), resulting in 11 function evaluations. According to Fig. 2, the MPP-based dimension-reduction method produces the best approximation of the limit-state function for this particular problem. The performance function using mean-point based dimension reduction method degenerates to a single point (see Fig. 2), leading to a null failure set.

Table 1 shows the results of the failure probability calculated by FORM, three variants of SORM due to Breitung [3], Hohenbichler et al. [4], and Cai and Elishakoff [5], mean-point-based univariate dimension-reduction method [2], proposed MPP-based univariate dimension-reduction method, and direct Monte-Carlo simulation using 10^6 samples. The MPP-based dimension-reduction method generates an excellent estimate of the failure probability. The dimension-reduction method using mean point, which yields poor approximation of the performance function (see Fig. 2), failed to provide a solution. Other commonly used reliability methods, such as FORM and SORM (all three variants), overpredicted the Monte-Carlo result by more than 200%. The SORM results are the same as the FORM results, indicating that there is no improvement over FORM for this highly nonlinear problem.

3.2. Example 2 – ten-bar truss structure

A ten-bar, linear-elastic, truss structure, as shown in Fig. 3, was studied in this example to examine the accuracy and efficiency of the proposed reliability method. The Young's modulus of the material is 10^7 psi. Two concentrated forces of 10^5 lb are applied at nodes 2 and 3, as shown in Fig. 3. The cross-section area X_i, $i = 1,\ldots,10$ for each bar is normally distributed random variable with mean $\mu = 2.5$ in^2 and standard deviation $\sigma = 0.5$ in^2. According to the loading condition, the maximum displacement $[(v_3(X_1,\ldots,X_{10})]$ occurs at node 3, where a permissible displacement is limited to 18 in. Hence, the limit-state function is:

$$g(\boldsymbol{X}) = 18 - v_3(X_1, \cdots, X_{10}) \tag{10}$$

From the MPP search involving finite-difference gradients, the reliability index is $\beta = \|\boldsymbol{v}^*\| = 1.3642$. Table 2 shows the failure probability of the truss, calculated using the proposed MPP-based univariate dimension-reduction method, mean-point based univariate

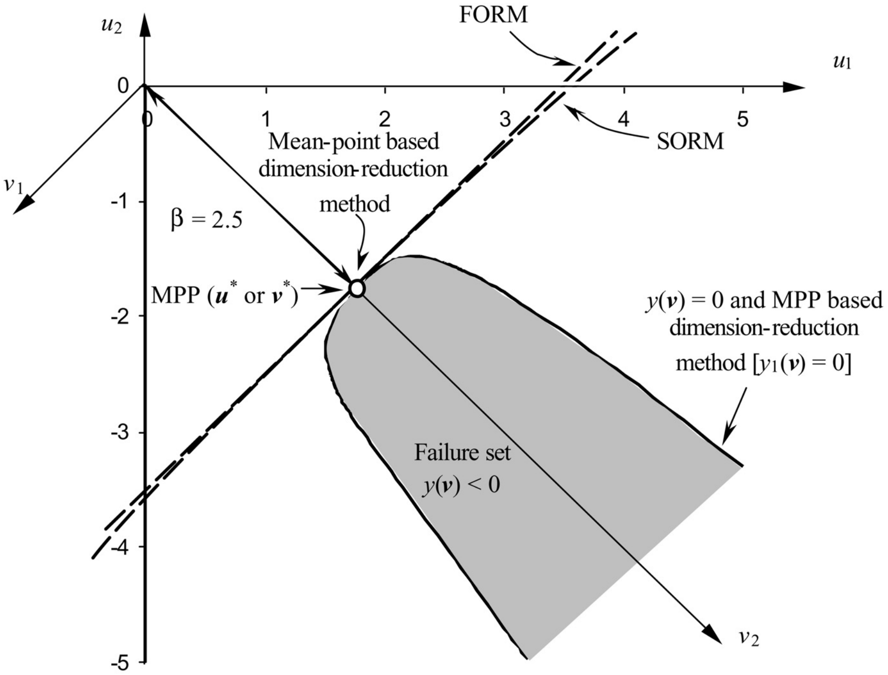

Fig. 2. Performance function approximations in Example 1.

Table 1
Failure probabilities for mathematical problem

Method	Failure probability	Number of function evaluations(a)
Mean-point-based univariate dimension-reduction [2]	–(b)	–(b)
MPP-based univariate dimension-reduction	0.002822	31(c)
FORM	0.006209	21
SORM (Breitung) [3]	0.006208	212
SORM (Hohenbichler et al.) [4]	0.006208	212
SORM (Cai and Elishakoff) [5]	0.006206	212
Direct Monte-Carlo simulation	0.002865	1,000,000

(a) Total number of times the original performance functions is calculated.
(b) Failed to provide a solution.
(c) $21 + (2 \times 5) = 31$.

dimension-reduction method [2], FORM, three variants of SORM [3,4,5], and direct Monte-Carlo simulation (10^6 samples). For the dimension-reduction method, five uniformly distributed points between $\mu-2\sigma$ and $\mu+2\sigma$ were deployed for function evaluations at each dimension. As can be seen from Table 2, both versions of the dimension-reduction method predict the failure probability more accurately than FORM and all three variants of SORM. This is because both dimension-reduction methods are able to approximate the performance function more accurately than FORM and SORM. A comparison of the number of function evaluations, also listed in Table 2, indicates that the mean-point-based dimension-reduction is the most efficient method. The number of function evaluations by the MPP-based dimension-reduction method is slightly larger than FORM but much less than SORM.

4. Conclusions

A new MPP-based univariate dimension-reduction method has been developed for predicting reliability of mechanical systems subject to random loads, material properties, and geometry. The method involves

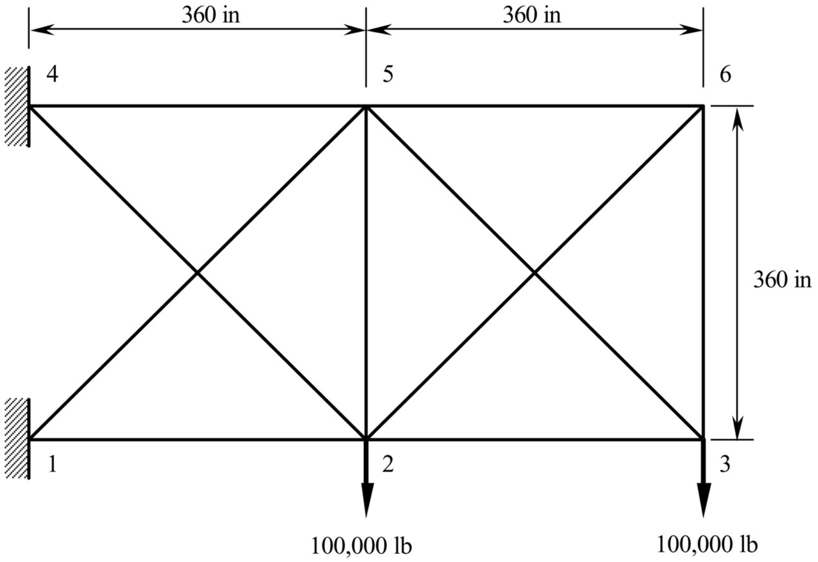

Fig. 3. A ten-bar truss structure.

Table 2
Failure probabilities for ten-bar truss

Method	Failure probability	Number of function evaluations[(a)]
Mean-point-based univariate dimension-reduction [2]	0.1364	51[(b)]
MPP-based univariate dimension-reduction	0.1431	177[(c)]
FORM	0.0863	127
SORM (Breitung) [3]	0.1286	506
SORM (Hohenbichler et al.) [4]	0.1524	506
SORM (Cai and Elishakoff) [5]	0.1467	506
Direct Monte-Carlo simulation	0.1397	1,000,000

(a) Total number of times the original performance functions is calculated.
(b) $(10 \times 5) + 1 = 51$.
(c) $127 + (10 \times 5) = 177$.

univariate function representation at MPP, approximate response surface generation using dimension reduction, and Monte Carlo simulation. The method yields higher-order approximation of a performance function without requiring any gradients. Two numerical examples involving an elementary mathematical function and a structural truss problem illustrate the proposed method. Results indicate that the proposed method provides accurate and computationally efficient estimates of the probability of failure.

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