On meshfree computations of shells

Sebastian Skatulla*, Carlo Sansour

School of Mechanical Engineering, The University of Adelaide, SA 5005, Australia

Abstract

In this paper we combine a meshfree moving least method (MLS) formulation with a Nitsche-like method to enforce essential boundary conditions. The formulation is based on a variational principle. The application in nonlinear structural mechanics involves the Green strain tensor and a hyperelastic material law. Various examples of shell deformations are presented which show the excellent performance of our proposed method.

Keywords: Shell analysis; Meshfree; Enforcement of boundary conditions

1. Introduction

One of the main reasons for the increase in research in the so-called *meshfree methods* [1,2] is the fact that these methods can deal especially well with problems which are characterized by large deformations, changing domain geometry, or necessitate higher order approximation consistency.

However, meshfree methods still struggle with some drawbacks and one of these is the fulfillment of essential boundary conditions. The explicit enforcement of essential boundary conditions, which is the common procedure in FEM, is not suitable for meshfree methods. This is rooted in the fact that the approximation functions do not possess the Kronecker–Delta property.

Inspired by the so-called *Nitsche method* [3] we describe a method which enforces the essential boundary conditions of an elliptic PDE by a combination of a modified variational principle and the penalty method. With regard to shell structures, we investigate the general applicability of the method as well as the dependency of the solution accuracy on the chosen penalty or stabilization parameter.

Our paper is structured as follows. After a brief introduction to the moving least square method, a Nitsche-like formulation is presented. The potential of the proposed method and its applicability to non-linear shell problems is revealed using two numerical simulations of shells. Hereby two different hyperelastic constitutive laws – the linear Saint-Venant-Kirchhoff and a non-linear statistically based model of Arruda and Boyce [4] are utilized.

2. Moving least square method

In the moving least square method (MLS) [5] an approximation for a solution is constructed based on a given set of particles. In the following we outline the MLS method briefly. Let us consider any function $u(x)$ defined over the field Ω. A possible approximation for $u(x)$ is defined by a complete polynomial $\mathbf{P}(x)$ and its non-constant coefficients $\mathbf{a}(x)$:

$$u^h(x) = \mathbf{P}\,(x) \cdot \mathbf{a}\,(x) \tag{1}$$

where scalar products of vectors are denoted by a dot. To each particle, a so-called weight function Φ with compact support is attached. ϱ defines the so-called *influence radius* of Φ. The sum of all particles with coordinates x_I, that support the point x, constitute the set Λ. With the help of this set a weighted least square fit in the vicinity of a point x can be constructed according to:

$$J(\mathbf{a}\,(\mathrm{x})) := \sum_{I\in\Lambda} [\mathbf{P}(x_{\mathrm{I}}) \cdot \mathbf{a}(x) - u(x_{\mathrm{I}})]^2 \, \Phi\left(\frac{x - x_I}{\varrho}\right) \tag{2}$$

The least square fit is weighted by a function Φ which, in our case, is taken to be a cubic spline.

The unknown coefficients $\mathbf{a}(\mathrm{x})$ can be determined by minimising the functional J with respect to $\mathbf{a}(x)$. Then the substitution of the coefficients $\mathbf{a}(x)$ in Eq. (1) provides the approximation of $u(x)$ as:

* Corresponding author. Tel.: +61 8 8303 5460; Fax: +61 8 8303 4367; E-mail: sebastian.skatulla@adelaide.edu.au

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

$$u^h(x) = \mathbf{P}(x) \cdot \mathbf{M}^{-1}(x) \sum_{I \in \Lambda} \mathbf{P}(x_I)\, \Phi\left(\frac{x - x_I}{\varrho}\right) u_I \tag{3}$$

where $\mathbf{M}(x)$ is the so-called *moment matrix* of the weight function Φ:

$$\mathbf{M}\,(x) = \sum_{I=\Lambda} \mathbf{P}(x_I)\, \mathbf{P}(x_I)\, \Phi\left(\frac{x - x_I}{\varrho}\right) \tag{4}$$

and u_I are the so-called particle parameters. Due to the continuity of the cubic splines the MLS approximation possesses at least C^2.

3. A Nitsche-like method

Let us consider a non-linear boundary value problem on domain Ω with boundary $\partial\Omega$. Dirichlet boundary conditions are prescribed on $\partial\Omega_D \subset \partial\Omega$ and Neumann boundary conditions are prescribed on $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$.

Now let $\mathbf{F}(\mathbf{u}) = \mathbf{1} + \text{Grad}\ \mathbf{u}$ be the deformation gradient and $\mathbf{E}(\mathbf{u}) = \mathbf{F}^T\mathbf{F}$, the Green strain tensor. Assume a hyperelastic material behaviour and let $\psi\,(\mathbf{E})$ define the stored energy function. Further, let $\mathcal{W}_{ext}$ define the external potential as follows;

$$\mathcal{W}_{ext}(\mathbf{u}) = -\int_\Omega \mathbf{f} \cdot \mathbf{u}\ \mathrm{dV} - \int_{\partial\Omega_N} \hat{\mathbf{t}} \cdot \mathbf{u}\ \mathrm{dA} \tag{5}$$

where $\mathbf{f}$ is the body force and $\hat{\mathbf{t}}$ is the external traction vector prescribed on Ω_N. We start from the following variational statement:

$$\delta \prod(\mathbf{u}) = \int_\Omega \mathbf{S} : \delta\mathbf{E}\ \mathrm{dV} - \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\ \mathrm{dV} - \int_{\partial\Omega_N} \hat{\mathbf{t}} \cdot \delta\mathbf{u}\ \mathrm{dA} = 0 \tag{6}$$

where $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor given by:

$$\mathbf{S}(\mathbf{u}) = \frac{\partial\psi\,(\mathbf{E})}{\partial\mathbf{E}} \tag{7}$$

The double dot operator (:) denotes the scalar product of tensors.

The above functional corresponds to the following Euler–Lagrange field equation:

$$\text{Div}(\mathbf{FS}) + \mathbf{f} = \mathbf{0}, \quad \text{on } \Omega, \qquad \mathbf{FS}\boldsymbol{\nu} - \hat{\mathbf{t}} = \mathbf{0}, \text{on } \partial\Omega_N, \qquad \mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \partial\Omega_D \tag{8}$$

where $\boldsymbol{\nu}$ defines the normal vector at the boundary.

To incorporate the essential boundary conditions in the functional itself, that is to enforce these conditions as a Euler–Lagrange equation, the functional (6) is modified in the following way:

$$\begin{aligned} \delta \prod(\mathbf{u}) \quad = \quad & \int_\Omega \mathbf{S} : \delta\mathbf{E}\ \mathrm{dV} - \int_{\partial\Omega_D} \mathbf{t} \cdot \delta\mathbf{u}\ \mathrm{dA} \\ & - \int_{\partial\Omega_D} \delta\mathbf{t} \cdot (\mathbf{u} - \hat{\mathbf{u}})\ \mathrm{dA} + \beta \int_{\partial\Omega_D} (\mathbf{u} - \hat{\mathbf{u}}) \cdot\ \delta\mathbf{u}\ \mathrm{dA} \\ & - \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\ \mathrm{dV} - \int_{\partial\Omega_N} \hat{\mathbf{t}} \cdot \delta\mathbf{u}\ \mathrm{dA} = 0 \end{aligned} \tag{9}$$

It is crucial to note that the relation holds $\mathbf{t} = \mathbf{FS}\boldsymbol{\nu}$. That is, the variation is to be considered with respect to $\mathbf{FS}\boldsymbol{\nu}$. The fourth term in Eq. (9) is a stabilization term together with the stabilization parameter β.

4. Numerical examples

In order to illustrate the impact of nonlinearity and material on the applicability and accuracy of a Nitsche-like formulation in shell analysis, two different hyperelastic material models, the linear Saint-Venant-Kirchhoff and the statistically-based constitutive model for rubber-like materials of Arruda and Boyce [4,6] have been investigated.

4.1. Pinched cylinder with free edges

Our first example is a classical one, a cylindrical shell with length $L = 10.35$ m, radius $R = 4.953$ m and thickness $h = 0.094$ m which is free at the edges. The shell is subjected to two vertically opposite point loads at its central points (points A). Assuming appropriate symmetry boundary conditions, the cylinder is modelled using one octant with 6 particles in longitudinal, 16 in radial, and 3 in thickness direction. As constitutive law the Saint-Venant-Kirchhoff model has been chosen. The material parameters are Young's modulus $E = 1.05 \times 10^4$ N/m^2 and Poisson's ratio $\nu = 0.3125$. The displacement diagram in Fig. 1 shows that the deformation

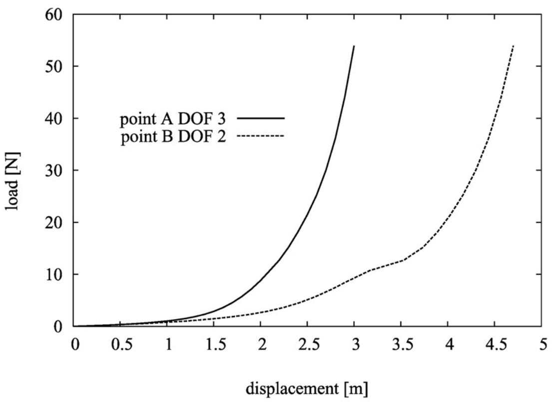

Fig. 1. Displacement diagram.

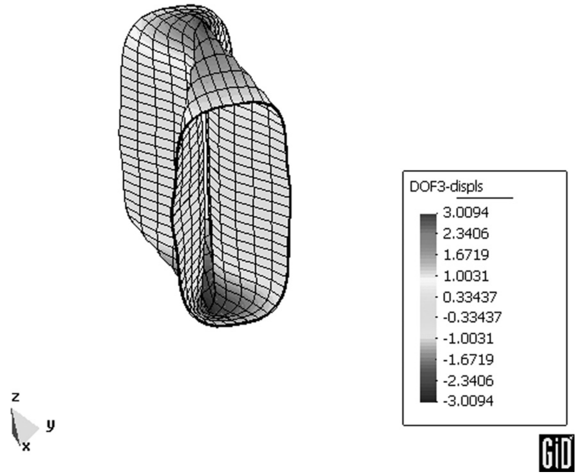

Fig. 2. Deformed configuration at loading parameter 53.95 N.

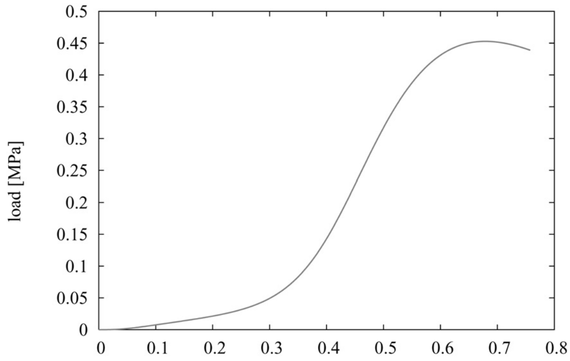

Fig. 3. Displacement diagram.

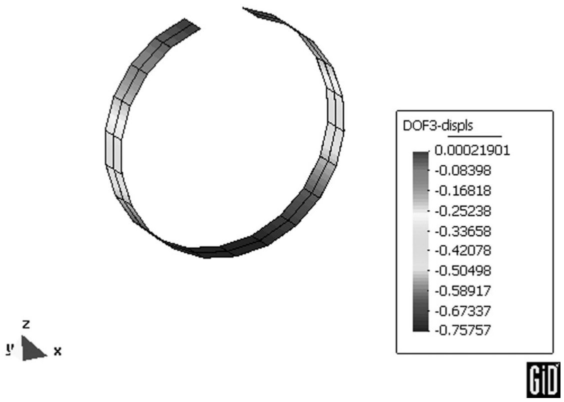

Fig. 4. Deformed configuration at loading parameter 0.439 MPa.

process is split into two parts. The first part is bending dominated which results in large deformations for small loading parameters. The second part is characterized by a steep slope. In Fig. 2, the final deformed configuration is displayed. It should be noted that this example has been considered by many authors using different shell finite elements. In fact, our numerical results are in good agreement with those reported in the literature.

4.2 Square sheet under pressure loading

Our second example, which utilizes a statistically based model for rubber material, is a square sheet with a length of $L = 0.2$ m and thickness $h = 0.0003$ m. The constitutive parameters involve three constants: a shear modulus $C_R = 1.56$ MPa, a bulk modulus $k = 1000$ MPa and a parameter $N = 8$, which addresses the limited extensibility of the macromolecular network structure of the rubber material. The sheet is fixed on two opposite edges and subjected to a constant pressure load on its top surface. Due to symmetry conditions, one half of the sheet is modelled using 11 particles in length, 3 in width and 3 in thickness directions. The deformation process is displayed in Fig. 3 and the deformed configuration is depicted in Fig. 4.

5. Conclusion

In this paper we demonstrated the excellent performance of a Nitsche-like formulation for a MLS approximation method within shell analysis. Generally, we could achieve excellent results already for low discretization levels. A linear calculation without the stabilization term provided meaningful results for both examples, but the stabilization term was needed in order to perform a nonlinear calculation. This behaviour is founded in the fact that essential boundary conditions are only satisfied to a high accuracy if a penalty-like stabilization is retained in the entire variational formulation. This is especially true in the case of the Saint-Venant-Kirchhoff model, which results, due to relatively high values of the material constants, in a stiffer tangent matrix. In such case the stabilization parameter must be set to a high value. However, in general, the formulation allows for low levels of penalty parameters without sacrificing accuracy. This would not be possible in a pure penalty formulation.

References

[1] Nayroles B, Touzot G, Villon P. Generalizing the finite element method: diffuse approximation and diffuse elements. Computational Mechanics 1992;10:307–318.

[2] Belytschko T, Lu YY, Gu L. Element free galerkin methods. International Journal for Numerical Methods in Engineering 1994;37:229–256.

[3] Nitsche J. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh Math Sem Univ Hamburg 1970–1971;36:9–15.

[4] Arruda EM, Boyce MC. A three-dimensional constitutive model for the large stretch behaviour of rubber elastic materials. Journal of the Mechanics and Physics of Solids 1993;41:389–412.

[5] Lancaster P, Salkauskas K. Surface generated by moving least square methods. Mathematics of Computations 1981;37(155):141–158.

[6] Sansour C, Feih S, Wagner W. On the performance of enhanced strain finite elements in large strain deformations of elastic shells. International Journal for Computer-Aided Engineering and Software 2003;20(7):875–895.