The local discontinuous Galerkin method and component design integration for 3D elasticity

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Abstract

In this paper, the focus is on the developments towards a local discontinuous Galerkin (LDG) method for 3D elasticity and its application to contemporary practical engineering design. On the theoretical front, the definition of the numerical flux that guarantees stability of the method, based on the energy identity of elasticity, is presented. Among the practical advantages of the LDG method is its ability to handle non-congruent meshes which is a very useful feature for designing complex structures. We demostrate this advantage via results of the analysis of an armored vehicle's barrel-breech system. Optimal convergence rates of the method are shown via numerical experiments using a P1 approximation, for congruent as well as non-congruent meshes.

Keywords: Local discontinuous Galerkin method; 3D elasticity; Non-congruent meshes

1. Introduction

The discontinuous Galerkin methods have been successfully used in solving fluid flow problems. Cockburn [1] has provided an exposition of a framework for the construction of DG methods based on the discrete energy identity for various problems. In this paper we show for the first time how this concept can be used in the context of 3D elasticity, resulting in an LDG formulation. We also show, via a numerical experiment, the ability of the method in handling non-congruent meshes.

2. The general DG method for the 3D elasticity

Of interest here is the problem of 3D elasticity governed by:

$$-\sigma_{ij,i} = f_j \quad \text{in} \quad \Omega \tag{1}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{in} \quad \Omega \tag{2}$$

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \quad \text{in} \quad \Omega \tag{3}$$

$$u_j = u_D \quad \text{in} \quad \partial\Omega^1 \tag{4}$$

$$\sigma_{ij}n_j = T_i \quad \text{in} \quad \partial\Omega^2,\ \partial\Omega^1 \cup \partial\Omega^2 = \partial\Omega \tag{5}$$

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The energy identity for the continuum can be derived from the above equations and can be shown to yield:

$$\int_\Omega C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \int_\Omega f_j U_j + \int_{\partial\Omega^1} \sigma_{ij} n_i u_D + \int_{\partial\Omega^2} T_i u_i \tag{6}$$

We establish a similar discrete energy identity for the DG method to follow.

The DG Method: We take $\sigma^h_{ij}|_\kappa$ and $\varepsilon^h_{kl}|_\kappa$ in the same space V_k and $U^h_i|_\kappa$ in the space U_k. The weak formulation and discrete energy identity for the DG method are given by the following equations:

$$\int_\kappa \sigma^h_{ij}\omega_{,i} - \int_{\partial\kappa} \hat{\sigma}^h_{ij} n_i \omega = \int_\kappa f_j \omega, \qquad \forall \omega \in U_k \tag{7}$$

$$\int_\kappa \sigma^h_{ij} v = \int_\kappa C_{ijkl}\varepsilon^h_{kl} v, \forall v \in V_k \tag{8}$$

$$\int_\kappa \varepsilon^h_{ij} v = -\frac{1}{2}\int_\kappa (u^h_k v_{,l} + u^h_l v_{,k}) + \int_{\partial\kappa} \frac{1}{2}(\hat{u}^h_k n_l + \hat{u}^h_l n_k) v \qquad \forall v \in V_k \tag{9}$$

$$\hat{u}^h_j = u_D \qquad \in \partial\Omega^1 \tag{10}$$

$$\hat{\sigma}^h_{ij} n_j = T_i \qquad \in \partial\Omega^2 \tag{11}$$

$$\sum_\kappa \int_\kappa \varepsilon^h_{kl}\sigma^h_{kl} + \Theta = \int_\Omega f_j u^h_j + \int_{\partial\Omega^1} \sigma^h_{ij} n_i u_D + \int_{\partial\Omega^2} T_i u^h_i \tag{12}$$

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$$\Theta = \sum_e \int_e \left(\left(\left\{ \sigma_{ij}^h \right\} - \hat{\sigma}_{ij}^h \right) \left[\!\left[n_i u_j^h \right]\!\right] + \left(\left\{ u_j^h \right\} - \hat{u}_j^h \right) \left[\!\left[n_i \sigma_{ij}^h \right]\!\right] \right) \tag{13}$$

$$+ \int_{\partial\Omega^1} \left(\sigma_{ij}^h - \hat{\sigma}_{ij}^h \right) n_i u_j + \int_{\partial\Omega^2} \sigma_{ij}^h n_i u_j - \hat{u}_j^h n_i \sigma_{ij}$$

For the stability of the method we must render Θ positive and the numerical fluxes are then given by:

$$\hat{\sigma}_{ij}^h = \{\sigma_{ij}\}^h - A_{ijkl}[\![n_k u_l^h]\!] - B_{ijk}[\![n_l \sigma_{lk}^h]\!] \qquad \forall \partial\kappa \in \partial\Omega^e \tag{14}$$

$$\hat{u}_j^h = \{u_j^h\} - C_{jk}[\![n_l \sigma_{lk}^h]\!] - D_{jkl}[\![n_k u_l^h]\!] \qquad \forall \partial\kappa \in \partial\Omega^e \tag{15}$$

$$\hat{u}_j^h = u_j^h \qquad \forall \partial\kappa \in \partial\Omega^2 \tag{16}$$

$$\hat{\sigma}_{ij}^h = \sigma_{ij}^h - \delta^2 n_i u_j \qquad \forall \partial\kappa \in \partial\Omega^1 \tag{17}$$

where

$$[n_i u_j^h] = n_i^+ u_j^{h+} + n_i^- u_j^{h-} \text{ and } \{a\} = \tfrac{1}{2}(a^+ + a^-)$$

and n^+ and n^- are the inward and outward normals at an element interface respectively.

Conditions on the constants for well posedness of the method:

- $B_{ijk} = -D_{kij}$
- A_{ijkl} must be positive definite

$C_{jk} \equiv 0$ yields the LDG method.

3. Numerical results

Test problem: to demonstrate the convergence rates we have solved the problem of a uniform cube, with a quadratic displacement field, consistent tractions and body forces. The problem has been solved employing a P1 formulation in the weakform described earlier. Figure 1 depicts the convergence results for both a congruent and non-congruent mesh along with the details of the non-congruent mesh. We observe an optimal convergence rate ($k + 1$) in the $\mathbf{L}^2$ norm in both cases.

A practical engineering problem: in order to demonstrate the advantage of the LDG method from an engineering design standpoint, we have analyzed a *barrel-breech* system that is part of an armored tank. The breech is clamped at its end and the barrel is subjected to uniform internal pressure along its entire length. Given the large size of such components, it is highly desirable for the designers at the mesh generation level to be able to mesh different parts of the system independently. The challenge then is to combine the different meshes obtained from disparate sources to analyze the integrated system design. The LDG method allows this sort of a fusion of independent designs. The barrel and the breech were separately meshed and the two non-congruent meshes are shown in Fig. 2. Figure 3 shows the details of the displacements at the *barrel-breech* interface for the design obtained from the above meshes. These results agree with the results obtained from a single mesh design (not shown here).

4. Conclusions

We have described the LDG method for 3D elasticity applications, the process of defining the fluxes for stability of the method and the numerical proof of optimal convergece rates for the method for congruent and non-congruent meshes. We have then demonstrated the advantage of using the LDG method in processing and synthesizing mesh data arising from different sources,

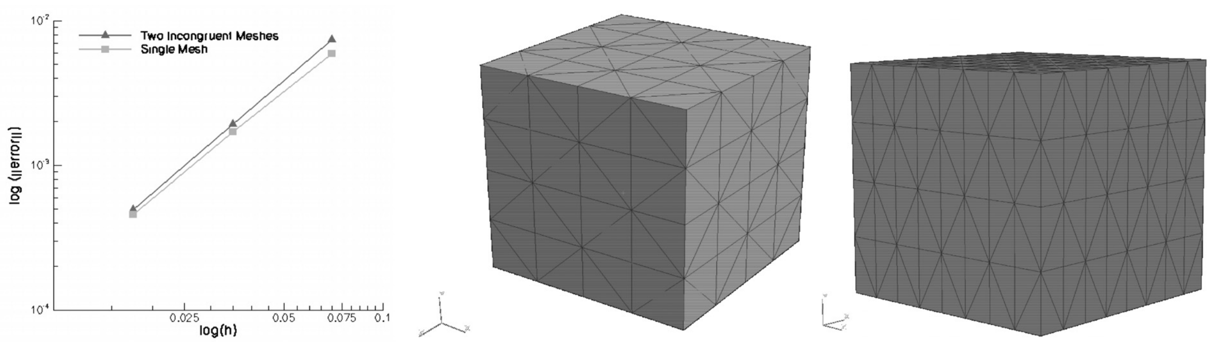

Fig. 1. Convergence results of the cube and the two halves of the cube with non-congruent mesh at their interface (congruent mesh not shown).

Fig. 2. Details of the non-congruent mesh at the breech-barrel interface.

Fig. 3. Details of displacement results at the breech-barrel interface.

which is very useful from a practical engineering design standpoint.

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