Crack in a homogeneous piezoelectric material bonded to a functionally graded piezoelectric half plane

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Abstract

The problem for a crack located within a homogeneous piezoelectric material bonded to a functionally graded piezoelectric material is studied. It is subjected both to the mechanical and electrical loads. By using the Fourier transform and residue theorem, the problem can be reduced to a system of singular integral equations and then solved by the Gauss–Chebyshev integration formula. The dependency of stress and electric displacement intensity factors on the material nonhomogeneity parameter is discussed.

Keywords: Functionally graded piezoelectric material; Crack; Stress and electric intensity factors; Gauss–Chebyshev integration formula

1. Introduction

Functionally graded material (FGM) is the material with continuously varying properties such as stiffness. It can efficiently remove the sharp change on the interface between two distinct materials and then reduce the interface failure possibility. The studies on the functionally graded elastic material become gradually mature. Typical papers include Delale et al. [1], Erdogan [2], Erdogan et al. [3], etc.

Piezoelectric materials have been widely used in electromechanical devices. Owing to the brittle properties of piezoelectric ceramics, it faces the similar problems that ever occurred in elastic ones. Recently, many researchers are interested in the application of piezoelectric material with continuously varying material properties [4,5]. They are called functionally graded piezoelectric material (FGPM). Li and Weng [6] first applied the concept of fracture mechanics on a finite crack in a strip of functionally graded piezoelectric material. Wang [7] solved the antiplane crack and collinear crack problems in FGPM. Ueda [8] obtained the solutions for a crack in FGPM strip bonded to two elastic surface layers.

In this paper, one FGPM bonded to a homogeneous piezoelectric half plane with an internal crack is studied. The material properties are supposed to be in exponential form and vary in the direction normal to the crack line. The problem can be reduced into a system of singular integral equations after applying the Fourier transform and solved numerically by using the Gauss–Chebyshev integration technique. The stress and electric displacement intensity factors are then obtained from the near electro-mechanical field solution.

2. Description of the problem

Consider a crack of length $2a_0$ embedded in a homogeneous half plane bonded to a FGPM, as shown in Fig. 1. Assuming that z-axis is the poling direction, the antiplane mechanical field and inplane electrical field are coupled. The constitutive equations can be written as:

$$\tau_{jz}^{I} = c_{44} w_{,j}^{I} + e_{15}\phi_{,j}^{I},\ D_j^{I} = e_{15} w_{,j}^{I} - \varepsilon_{11}\phi_{,j}^{I} \tag{1a}$$

for material I, and

$$\tau_{jz}^{II} = c_{44}(x) w_{,j}^{II} + e_{15}(x)\phi_{,j}^{II},\ D_j^{II} = e_{15}(x) w_{,j}^{II} - \varepsilon_{11}(x)\phi_{,j}^{II} \tag{1b}$$

for material II. In Eq. (1), τ_{jz}, w, D_j, and ϕ ($j = x, y$) are the shear stresses, antiplane displacement, inplane electric displacements and electric potential, respectively. The material constants $c_{44}(x)$, $e_{15}(x)$, $\varepsilon_{11}(x)$ called the

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shear modulus, piezoelectric constant and dielectric constants, respectively, are assumed to vary along the x-direction (normal to the crack line) as:

$$c_{44}(x) = c_{44}\exp(\gamma x), e_{15}(x) = e_{15}\exp(\gamma x), \varepsilon_{11}(x) = \varepsilon_{11}\exp(\gamma x) \quad \text{for } x < 0 \tag{2}$$

where γ is called nonhomogeneous parameter. The constants c_{44}, e_{15}, and ε_{11} are the material properties of homogeneous material. The equilibrium equations can thus be written as:

$$\begin{cases} c_{44}\nabla^2 w^I + e_{15}\nabla^2\phi^I = 0 \\ e_{15}\nabla^2 w^I - \varepsilon_{11}\nabla^2\phi^I = 0 \end{cases} \tag{3a}$$

$$\begin{cases} c_{44}\nabla^2 w^{II} + e_{15}\nabla^2\phi^{II} + \gamma\left(c_{44}w^{II}_{,x} + e_{15}\phi^{II}_{,x}\right) = 0 \\ e_{15}\nabla^2 w^{II} - \varepsilon_{11}\nabla^2\phi^{II} + \gamma\left(e_{15}w^{II}_{,x} - \varepsilon_{11}\phi^{II}_{,x}\right) = 0 \end{cases} \tag{3b}$$

The solution of Eq. (3) can be expressed as [2]:

$$\begin{cases} w^I(x,y) = \frac{1}{2\pi}\int_\infty^\infty f_{11}(\alpha,y)e^{-i\alpha x}d\alpha + \frac{2}{\pi}\int_0^\infty g_{11}(x,\alpha)\sin(\alpha y)d\alpha \\ \phi^I(x,y) = \frac{1}{2\pi}\int_\infty^\infty f_{21}(\alpha,y)e^{-i\alpha x}d\alpha + \frac{2}{\pi}\int_0^\infty g_{21}(x,\alpha)\sin(\alpha y)d\alpha \end{cases} \tag{4a}$$

$$\begin{cases} w^{II}(x,y) = \frac{2}{\pi}\int_0^\infty g_{12}(x,\alpha)\sin(\alpha y)d\alpha \\ \phi^{II}(x,y) = \frac{2}{\pi}\int_0^\infty g_{22}(x,\alpha)\sin(\alpha y)d\alpha \end{cases} \tag{4b}$$

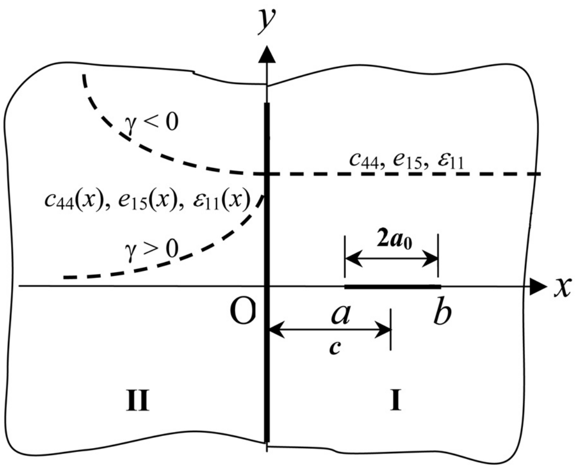

Fig. 1. A crack in a homogeneous piezoelectric material bonded to a functionally graded piezoelectric half plane.

In this paper, both impermeable and permeable crack conditions are considered. It is assumed that both the traction and electric displacement are exerted on the crack surface. Only the upper half plane ($y > 0$) is considered due to the symmetry of the problem. To satisfy the regularity conditions, both w and ϕ must disappear when x and y approach infinite. The undetermined functions are:

$$\begin{cases} f_{11}(\alpha,y) = A_1(\alpha)\exp(-\alpha y) \\ f_{21}(\alpha,y) = B_1(\alpha)\exp(-\alpha y) \\ g_{11}(x,\alpha) = C_1(\alpha)\exp(-\alpha x) \\ g_{21}(x,\alpha) = D_1(\alpha)\exp(-\alpha x) \end{cases} \tag{5a}$$

$$\begin{cases} g_{12}(x,\alpha) = E_2(\alpha)\exp\left(\frac{-\gamma+\sqrt{\gamma^2+4\alpha^2}}{2}x\right) \\ g_{22}(x,\alpha) = F_2(\alpha)\exp\left(\frac{-\gamma+\sqrt{\gamma^2+4\alpha^2}}{2}x\right) \end{cases} \tag{5b}$$

where $A_1(\alpha)$, $B_1(\alpha)$, $C_1(\alpha)$, $D_1(\alpha)$, $E_2(\alpha)$ and $F_2(\alpha)$ depend on the following boundary conditions:

$$w^I(0,y) = w^{II}(0,y) \tag{6a}$$

$$\phi^I(0,y) = \phi^{II}(0,y) \tag{6b}$$

$$\tau^I_{xz}(0,y) = \tau^{II}_{xz}(0,y) \tag{6c}$$

$$D^I_x(0,y) = D^{II}_x(0,y) \tag{6d}$$

$$w^I(x,0) = 0 \quad \text{for } 0 \le x < a \text{ and } b < x < \infty \tag{6e}$$

$$\phi^I(x,0) = 0 \quad \text{for } 0 \le x < a \text{ and } b < x < \infty \tag{6f}$$

$$w^{II}(x,0) = 0 \text{ for } -\infty \le x < 0 \tag{6g}$$

$$\phi^{II}(x,0) = 0 \text{ for } -\infty \le x < 0 \tag{6h}$$

(i) impermeable crack

$$\tau^I_{yz}(x,0) = \tau(x) \quad \text{for } a \le x \le b \tag{7a}$$

$$D^I_y(x,0) = D(x) \quad \text{for } a \le x \le b \tag{7b}$$

(ii) permeable crack

$$\tau^I_{yz}(x,0) = \tau(x) \quad \text{for } a \le x \le b \tag{8a}$$

$$\phi^I(x,0) = 0 \quad \text{for } a \le x \le b \tag{8b}$$

$$D^I_y(x,0) = D_c(x,0) = D(x) \quad \text{for } a \le x \le b \tag{8c}$$

3. Analysis method

After applying the boundary conditions Eqs. (6a)–(6d) and then taking Fourier inverse transform, four unknown functions can be expressed by the remaining two functions $A_1(\alpha)$ and $B_1(\alpha)$ as:

$$\begin{cases} E_2(\alpha) - C_1(\alpha) = \frac{1}{2\pi}\int_\infty^\infty \left(\frac{\alpha}{\alpha^2+\rho^2}\right)(i\rho)A_1(\rho)d\rho \\ F_2(\alpha) - D_1(\alpha) = \frac{1}{2\pi}\int_\infty^\infty \left(\frac{\alpha}{\alpha^2+\rho^2}\right)(i\rho)B_1(\rho)d\rho \\ c_{44}(sE_2(\alpha) + \alpha C_1(\alpha)) + e_{15}(sF_2(\alpha) + \alpha D_1(\alpha)) \\ \quad = \frac{-1}{2\pi}\int_\infty^\infty \left(\frac{\alpha}{\alpha^2+\rho^2}\right)(i\rho)\left[c_{44}A_1(\rho) + e_{15}B_1(\rho)\right]d\rho \\ e_{15}(sE_2(\alpha) + \alpha C_1(\alpha)) - \varepsilon_{11}(sF_2(\alpha) + \alpha D_1(\alpha)) \\ \quad = \frac{1}{2\pi}\int_\infty^\infty \left(\frac{\alpha}{\alpha^2+\rho^2}\right)(i\rho)\left[-e_{15}A_1(\rho) + \varepsilon_{11}B_1(\rho)\right]d\rho \end{cases} \tag{9}$$

where $s = \alpha_2 - \gamma/2$ and $\alpha_2 = \sqrt{\alpha^2 + \gamma^2/4}$

3.1. Impermeable crack

We can define two dislocation functions $g_1(x)$ and $g_2(x)$ as:

$$g_1(x) = \frac{\partial}{\partial x} w_1(x, 0) \tag{10a}$$

$$g_2(x) = \frac{\partial}{\partial x} \phi_1(x, 0) \tag{10b}$$

They must satisfy the following equations to validate Eqs. (6e) and (6f):

$$\int_a^b g_1(t)dt = \int_a^b g_2(t)dt = 0 \tag{11}$$

The two remaining unknown functions can be solved as:

$$\begin{aligned} A_1(\alpha) &= \frac{i}{\alpha} \int_a^b g_1(t) e^{i\alpha t} dt \\ B_1(\alpha) &= \frac{i}{\alpha} \int_a^b g_2(t) e^{i\alpha t} dt \end{aligned} \tag{12}$$

Substituting Eq. (4a) into Eqs. (10a) and (10b) and using the residue theorem, the four undetermined functions can thus be solved:

$$C_1(\alpha) = E_2(\alpha) = \frac{(s-\alpha)}{2(-\alpha - s)} \int_a^b g_1(t) e^{-\alpha t} dt \tag{13}$$

$$D_1(\alpha) = F_2(\alpha) = \frac{(s-\alpha)}{2(-\alpha - s)\alpha} \int_a^b g_2(t) e^{-\alpha t} dt \tag{14}$$

Therefore, Eqs. (7a) and (7b) can be rearranged as follows:

$$\begin{aligned} \tau_{yz}^I(x,0) = \tau(x) &= c_{44} \frac{1}{\pi} \int_a^b \left[k_1(x,t) + k_2(x,t) \right] g_1(t) dt \\ &+ e_{15} \frac{1}{\pi} \int_a^b \left[k_1(x,t) + k_2(x,t) \right] g_2(t) dt \end{aligned} \tag{15}$$

$$\begin{aligned} D_y^I(x,0) = D(x) &= e_{15} \frac{1}{\pi} \int_a^b \left[k_1(x,t) + k_2(x,t) \right] g_1(t) dt \\ &- \varepsilon_{11} \frac{1}{\pi} \int_a^b \left[k_1(x,t) + k_2(x,t) \right] g_2(t) dt \end{aligned} \tag{16}$$

where:

$$k_1(x,t) = \frac{i}{2} \int_\infty^\infty -e^{i\alpha(t-x)} d\alpha \quad \text{and}$$

$$k_2(x,t) = \int_\infty^\infty \frac{\alpha - \alpha_2 + \gamma/2}{\alpha_2 + \alpha - \gamma/2} e^{-(t+x)\alpha} d\alpha$$

Rewrite the singular form of the kernels $k_1(x,t)$, Eqs. (15) and (16) become:

$$\begin{aligned} \tau_{yz}^I(x,0) = \tau(x) &= c_0 \frac{1}{\pi} \int_a^b \left[\frac{1}{t-x} + k_2(x,t) \right] g_1(t) dt \\ &+ e_0 \frac{1}{\pi} \int_a^b \left[\frac{1}{t-x} + k_2(x,t) \right] g_2(t) dt \end{aligned} \tag{17}$$

$$\begin{aligned} D_y^I(x,0) = D(x) &= e_0 \frac{1}{\pi} \int_a^b \left[\frac{1}{t-x} + k_2(x,t) \right] g_1(t) dt \\ &- \varepsilon_0 \frac{1}{\pi} \int_a^b \left[\frac{1}{t-x} + k_2(x,t) \right] g_2(t) dt \end{aligned} \tag{18}$$

The solutions of the singular integral equation with the Cauchy-type kernel have the form as:

$$g_i(t) = \frac{G_i(t)}{\sqrt{(t-a)(b-t)}}, i = 1, 2 \tag{19}$$

where $G_i(t)$ are bounded functions. The stress and electric displacement intensity factors can be derived as:

$$\begin{aligned} k_3(b) &= \lim_{x \to b^+} \sqrt{2(x-b)} \tau_{yz}^I(x,0) \\ &= -c_{44} \frac{G_1(b)}{\sqrt{(b-a)/2}} - e_{15} \frac{G_2(b)}{\sqrt{(b-a)/2}} \end{aligned} \tag{20}$$

$$\begin{aligned} k_3(a) &= \lim_{x \to a^-} \sqrt{2(a-x)} \tau_{yz}^I(x,0) \\ &= c_{44} \frac{G_1(a)}{\sqrt{(b-a)/2}} + e_{15} \frac{G_2(a)}{\sqrt{(b-a)/2}} \end{aligned} \tag{21}$$

$$\begin{aligned} k_3^D(b) &= \lim_{x \to b^+} \sqrt{2(x-b)} D_y^I(x,0) \\ &= -e_{15} \frac{G_1(b)}{\sqrt{(b-a)/2}} + \varepsilon_{11} \frac{G_2(b)}{\sqrt{(b-a)/2}} \end{aligned} \tag{22}$$

$$\begin{aligned} k_3^D(a) &= \lim_{x \to a^-} \sqrt{2(a-x)} D_y^I(x,0) \\ &= e_{15} \frac{G_1(a)}{\sqrt{(b-a)/2}} - \varepsilon_{11} \frac{G_2(a)}{\sqrt{(b-a)/2}} \end{aligned} \tag{23}$$

By employing the Gauss–Chebyshev integration formula, Eqs. (10), (17) and (18) can be reduced into the following $2n$ linear algebraic equations:

$$\begin{cases} \tau(x_r) = c_{44} \sum\limits_{k=1}^n \frac{1}{n} F_1(t_k) \left[\frac{1}{t_k - x_r} + \pi k_2(a_0 x_r + c, a_0 t_k + c) \right] \\ \qquad + e_{15} \sum\limits_{k=1}^n \frac{1}{n} F_2(t_k) \left[\frac{1}{t_k - x_r} + \pi k_2(a_0 x_r + c, a_0 t_k + c) \right] \\ D(x_r) = e_{15} \sum\limits_{k=1}^n \frac{1}{n} F_1(t_k) \left[\frac{1}{t_k - x_r} + \pi k_2(a_0 x_r + c, a_0 t_k + c) \right] \\ \qquad - \varepsilon_{11} \sum\limits_{k=1}^n \frac{1}{n} F_2(t_k) \left[\frac{1}{t_k - x_r} + \pi k_2(a_0 x_r + c, a_0 t_k + c) \right] \\ \sum\limits_{k=1}^n \frac{\pi}{n} F_1(t_k) = 0 \\ \sum\limits_{k=1}^n \frac{\pi}{n} F_2(t_k) = 0 \end{cases} \tag{24}$$

where $t_k = \cos(2k-1)\pi/2n$, $k = 1,2,\ldots,n$; $x_r = \cos r\pi/n$, $r = 1,2,\ldots,n-1$ are the nodes that satisfy Chebyshev polynomial of the first and second kind respectively. Then the intensity factors can be expressed as different forms as follows:

$$k_3(b) = -c_{44}\sqrt{a_0}F_1(1) - e_{15}\sqrt{a_0}F_2(1) \tag{25}$$

$$k_3(a) = c_{44}\sqrt{a_0}F_1(-1) + e_{15}\sqrt{a_0}F_2(-1) \tag{26}$$

$$k_3^D(b) = -e_{15}\sqrt{a_0}F_1(1) + \varepsilon_{11}\sqrt{a_0}F_2(1) \tag{27}$$

$$k_3^D(a) = e_{15}\sqrt{a_0}F_1(-1) - \varepsilon_{11}\sqrt{a_0}F_2(-1) \tag{28}$$

Here the unknown values of $F_i(1)$ and $F_i(-1)$, $(i = 1,2)$ can be obtained from the quadratic extrapolation from $F_i(n-1)$, $F_i(n-2)$, $F_i(n-3)$ and $F_i(2)$, $F_i(3)$, $F_i(4)$, respectively. It can be seen that the intensity factors depend on the mechanical loads and the electric loads.

3.2. Permeable crack

Only function $g_1(x)$ must be defined due to the condition (8b). The corresponding stress and electric displacement can be expressed as:

$$\tau_{yz}^I(x,0) = \tau(x) = c_0\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x} + k_2(x,t)\right]g_1(t)dt \tag{29}$$

$$D_y^I(x,0) = D(x) = e_0\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x} + k_2(x,t)\right]g_1(t)dt \tag{30}$$

Therefore the stress and electric displacement intensity factors are:

$$k_3(b) = -c_{44}\sqrt{a_0}F_1(1) \tag{31}$$

$$k_3(a) = c_{44}\sqrt{a_0}F_1(-1) \tag{32}$$

$$k_3^D(b) = -e_{15}\sqrt{a_0}F_1(1) \tag{33}$$

$$k_3^D(a) = e_{15}\sqrt{a_0}F_1(-1) \tag{34}$$

It can be seen that the intensity factors depend only on the mechanical loads.

4. Results and discussions

In the following discussions we take PZT-4 as the base material. The material properties are such as $c_{44} = 25.6$ GPa, $e_{15} = 12.7$ C/m^2, and $\epsilon_{11} = 6.46 \times 10^{-9}$ C/Vm. The stress and electric displacement intensity factors are normalized as:

$$k_i = \frac{k_3(i)}{\tau_0\sqrt{a_0}} = \frac{k_3^D(i)}{D_0\sqrt{a_0}}, i = a, b \tag{35}$$

In Eq. (24), we use $\tau_0 = 4.2$ MPa and $D_0 = 0.002$ C/m^2 as the uniform shear stress and electric displacement applied on the crack surfaces. Figure 2 plots the variations of normalized intensity factors with nonhomogeneous parameter γ of material II when $a_0 = 2/3$ cm. For the case $\gamma > 0$, the material properties of material II are weaker than those of the homogeneous material I. It is expected that the intensity factors at crack tip a are greater than those at crack tip b. For the homogeneous case (i.e. $\gamma = 0$), the problem becomes an infinite piezoelectric medium containing a crack. The normalized intensity factors should be equal to unity, i.e. $k_a = k_b = 1$. For the case when $\gamma < 0$, the material properties of material II are stronger and the intensity factors at crack tip b become larger. The magnitudes of the factors at crack tips a and b will gradually approach same values when the crack is far away from the interface. For the limiting case $\gamma \to \infty$, the problem is reduced to a crack approaching a traction-free and electrical-opened boundary surface. The corresponding kernel becomes:

$$k_2(x,t) = \frac{1}{t+x} \tag{36}$$

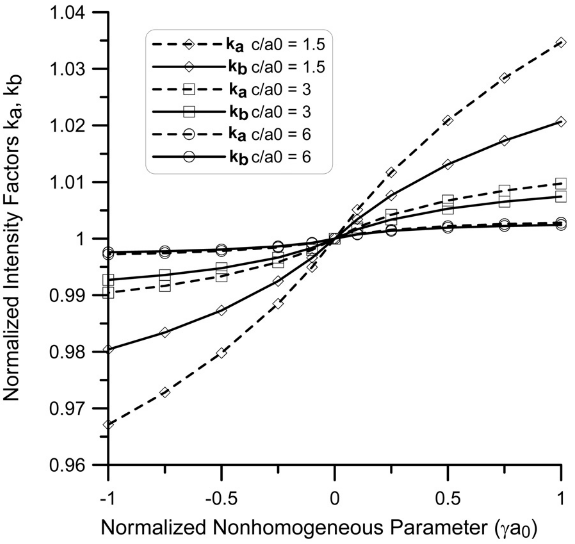

Fig. 2. Variations of normalized intensity factors with γ at different crack locations.

For the other limiting case $\gamma \to -\infty$ the interface becomes a clamped and electrical closed surface. The corresponding kernel is:

$$k_2(x,t) = \frac{-1}{t+x} \tag{37}$$

The results of Eqs. (36) and (37) can be found in previous studies.

5. Conclusions

The fracture behavior of an internal crack in a homogeneous material bonded to a functionally graded piezoelectric half plane has been studied. Both impermeable and permeable cases are considered. The results show that the normalized intensity factors are greater at the crack tip where the material properties are relatively stronger. Also, the stress and electric displacement intensity factors depend on the applied mechanical and electric loads for an impermeable crack and depend only on the mechanical loads for a permeable crack.

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