A formulation for evaluation of uncertain response due to multiple uncertain material properties in in-plane and plate structures

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Abstract

In the majority of research works in the stochastic finite element analysis, the main concern is put on the uncertainty in elastic modulus. This is due mainly to the importance of this parameter in stochastic response of structures but due also partly to the difficulties in introducing other material or geometrical uncertain parameters in the formulation. In this paper, a formulation to determine the statistical behavior due to multiple uncertain material parameters in in-plane and plate structures is given. To demonstrate the behavior of the proposed formulation, some examples are chosen and the results are compared with those obtained by means of classical Monte-Carlo simulation.

Keywords: Uncertain material parameters; Constitutive relationship; Response variability; Monte-Carlo simulation

1. Introduction

One of the features of structural materials is the intrinsic uncertainty in its mechanical properties which is spatially distributed over the system domain with a certain probabilistic characteristic. The uncertainties can be modeled as random process, which is defined as a parametered family of random variables: $\{X(\mathbf{x}){:}\mathbf{x} \in \Omega\}$, where Ω denotes the system domain. If the concern is on the uncertainties of multiple mechanical constants, such as $\{X(\mathbf{x}), Y(\mathbf{x}), \ldots{:}\ \mathbf{x} \in \Omega\}$, not only the statistical properties of respective random variables but also the cross-correlation between these variables have to be defined. Though many branches of concern exist in the literature, it is not too much to say that the studies in the area of stochastic finite element (FE) analysis have main concern on the effect of uncertain elastic modulus on the structural response variability [1,2]. Even though some indirect ways are suggested to include the Poisson ratio in the stochastic FE analysis [3], it is fairly recent work [4] where the *sole* effect of this parameter on the response variability is explicitly examined. In this paper, adopting the scheme of [4], a formulation to evaluate the response statistics due to multiple uncertain material parameters, in the context of weighted integral method, is presented.

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Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

2. Constitutive matrix with uncertainty

The constitutive relation is a function of material and geometrical parameters such as elastic modulus, Poisson ratio and thickness:

$$\mathbf{D}_\varepsilon, \mathbf{D}_\sigma = F(E,\nu), \mathbf{D}_b, \mathbf{D}_s = F(E,\nu,t,\kappa) \tag{1}$$

where $\mathbf{D}_\varepsilon$, $\mathbf{D}_\sigma$ denote constitutive matrices for plane strain and plane stress state, and $\mathbf{D}_b$, $\mathbf{D}_s$ stand for those of bending and shear part in plate finite element. E, ν, t and κ are elastic modulus, Poisson ratio, thickness of plate and shear correction factor, respectively. In this study, two material parameters, E and ν, are taken as uncertain parameters.

Following the development similar to [4], the constitutive matrix can be written as follows irrespective of the kinds of finite element under consideration:

$$\mathbf{D}(f_E, f_\nu) = \mathbf{D}_{\text{det}} + \bar{d}\sum_{k=0}^{\infty} f^{(k)}\mathbf{D}^{(k)} \tag{2}$$

where $\bar{d}$ are self-explanatory specific deterministic constants and the deterministic constitutive matrix is designated as $\mathbf{D}_{\text{det}}$. The k-th modified random field and sub-constitutive matrices are designated as $f^{(k)}$, $\mathbf{D}^{(k)}$. In deriving Eq. (2), mathematical expressions in Eq. (3) are employed:

$$E(\mathbf{x}) = E_o(1 + f_E(\mathbf{x})),\ \ \nu(\mathbf{x}) = \nu_o(1 + f_\nu(\mathbf{x})),\ \ \mathbf{x} \in \Omega \tag{3}$$

where E_o, v_o are the mean values of the elastic modulus and Poisson ratio, and $f_E(\mathbf{x})$, $f_v(\mathbf{x})$ are zero-mean, homogeneous random fields with standard deviation of σ_{EE}, σ_{vv} respectively. The symbol Ω denotes the structural domain to which the position vector $\mathbf{x}$ belongs.

3. Element stiffness with stochasticity

With the direct substitution of Eq. (2) into the equation for element stiffness matrix, $\mathbf{k} = \int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, dV$, the stochastic element stiffness matrix is written as:

$$\begin{aligned}\mathbf{k} &= \mathbf{k}_{\det} + \sum_{k=0}^{\infty} \delta\mathbf{k}^{(k)}\ (:= \Delta\mathbf{k}) \\ &= \int_V \mathbf{B}^T\mathbf{D}_{\det}\mathbf{B}\, dV + \bar{d}\sum_{k=0}^{\infty}\int_V f^{(k)}(\mathbf{x})\mathbf{B}^T\mathbf{D}^{(k)}\mathbf{B}\, dV\end{aligned} \tag{4}$$

where $\mathbf{k}_{\det}$ is deterministic stiffness and $\Delta\mathbf{k}$ is a deviatoric stiffness consisting of infinite sub-stochastic element stiffness.

3.1. Random variables in stochastic stiffness matrix

If we follow the precedent research work [1], where the strain-displacement matrix $\mathbf{B}$ is decomposed into a sum of constant matrix $\mathbf{B}_i$ multiplied by an independent polynomial p_i, $\mathbf{B} = \Sigma_{i=1}^{N_p}\mathbf{B}_i p_i$, the deviatoric stiffness $\Delta\mathbf{k}$ is given as a function of random variable:

$$\Delta\mathbf{k} = \bar{\mathbf{k}}_{ij}^{(k)} X_{ij}^{(k)},\ i,j = 1,2,\cdots,N_p \text{ and } k = 0,1,\cdots,\infty \tag{5}$$

with the brief notations of $\bar{\mathbf{k}}_{ij}^{(k)} = \bar{\mathrm{d}}\mathbf{B}_{\mathrm{i}}^{\mathrm{T}}\mathbf{D}^{(\mathrm{k})}\mathbf{B}_{\mathrm{j}}$ and $X_{ij}^{(k)} = \int_V f^{(k)}(\mathbf{x})p_i p_j\, dV$. As a consequence, not only the element stiffness matrix but also the global stiffness matrix and displacement vector are also given as functions of random variables.

3.2. Number of random variables

As can be deduced from the element stochastic stiffness matrix, (4) and (5), the total number of random variable is evaluated as:

$$N_{RV} = \sum_{k=0}^{m} N_{RV}^{(k)} = (m+1)\frac{1}{2}N_p(N_p+1) \tag{6}$$

if we truncate the expansion order in Eq. (5) to m.

4. Response statistics

Noting that m-kinds of random variables are involved in the stochastic stiffness matrix, the first order Taylor's expression for displacement vector can be written as follows:

$$\begin{aligned}\mathbf{U} &\approx \mathbf{U}_o - \sum_{k=0}^{m}\left\{\sum_{e=1}^{N_e}\sum_{RV=1}^{N_{RV}^{(k)}}\left(X_{RV}^{e(k)} - X_{RV}^{e(k)o}\right)\mathbf{K}_o^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV}^{e(k)}}\right]_E \mathbf{U}_o\right\} \\ &= \mathbf{U}_o - \Sigma_{RV} + \sum_{k=1}^{m}\bar{\Sigma}_{(k)}\end{aligned} \tag{7}$$

where sub- and superscript 'o', subscript 'E', and N_e denote mean value, evaluation at the mean and total number of finite element, respectively.

4.1. The first two moments

With Eq. (7), the mean and covariance of displacement are obtained as:

$$E[\mathbf{U}] = \mathbf{U}_o \tag{8}$$

$$Cov[\mathbf{U},\mathbf{U}] = E\left[\Delta\mathbf{U}\Delta\mathbf{U}^T\right] \tag{9}$$

With $\Delta\mathbf{U} = \mathbf{U} - E[\mathbf{U}]$ from Eq. (7), and after some manipulations, the covariance of response can be rearranged as:

$$Cov[\mathbf{U},\mathbf{U}] = \sum_{ei,ej=1}^{N_e}\mathbf{K}_o^{-1}\bar{\bar{\mathbf{F}}}_{eiej,E}\mathbf{K}_o^{-T} - \sum_{p=1}^{m}\sum_{q=1}^{m}E\left[\bar{\Sigma}_{(p)}\bar{\Sigma}_{(q)}^T\right] \tag{10}$$

4.2. Force-equivalent covariance matrix

The term, $\bar{\bar{\mathbf{F}}}_{eiej,E}$, named as 'force-equivalent covariance matrix', is given as an expectation on the force-equivalent terms as:

$$\bar{\bar{\mathbf{F}}}_{eiej,E} = E\left[\Delta\mathbf{k}^{ei}\mathbf{U}_o\mathbf{U}_o^T\Delta\mathbf{k}^{ej}\right] \tag{11}$$

Substituting the deviatoric stiffness in Eq. (4), $\bar{\bar{\mathbf{F}}}_{eiej,E}$ becomes:

$$\begin{aligned}\bar{\bar{\mathbf{F}}}_{eiej,E} &= E\Big[\left(\delta\mathbf{k}_{ei}^{(0)} + \delta\mathbf{k}_{ei}^{(1)} + \cdots + \delta\mathbf{k}_{ei}^{(m)}\right)\mathbf{U}_o\mathbf{U}_o^T \\ &\left(\delta\mathbf{k}_{ej}^{(0)} + \delta\mathbf{k}_{ej}^{(1)} + \cdots + \delta\mathbf{k}_{ej}^{(m)}\right)\Big] = \int_{V^{ei}}\int_{V^{ej}}\sum_{k=0}^{m}\sum_{l=0}^{m} \\ &\left\{E\left[f^{(k)}(\mathbf{x}_{ei})f^{(l)}(\mathbf{x}_{ej})\right]\tilde{\mathbf{k}}_{ei}^{(k)}\mathbf{U}_o\mathbf{U}_o^T\tilde{\mathbf{k}}_{ej}^{(l)}\right\}dV^{ej}V^{ei}\end{aligned} \tag{12}$$

where $\tilde{\mathbf{k}}_{ei}^{(k)} = \mathbf{B}_{ei}^T\mathbf{D}_{ei}^{(k)}\mathbf{B}_{ei}$. The expectation on the stochastic field function in Eq. (12) can be replaced with $R_{\alpha\beta}^{(kl)}(\xi_{ii},\xi_{ij},\xi_{jj})$ by means of general formula for random variables in multiplied form [4]. Here, it has to be mentioned that the transformation holds only for Gaussian variables, therefore it is implied in this study that the random material parameters follow the Gaussian distribution.

5. Numerical examples

The auto- and cross-correlation functions are assumed as follows:

$$R_{pp}(\xi) = \sigma_{pp}^2 \exp\left\{ -\frac{|\xi_1| + |\xi_2|}{d} \right\}, \quad p = E \text{ or } \nu$$
$$R_{\nu E}(\xi) = \rho_{\nu E} R_{\nu\nu}(\xi) \overset{or}{=} \rho_{\nu E} R_{EE}(\xi), \quad -1.0 \leq \rho_{\nu E} \leq 1.0 \qquad (13)$$

where d is a correlation distance and the cross-factor correlation (CCF) is designated as $\rho_{\nu E}$.

5.1. Classical Monte-Carlo simulation

To validate the performance of the proposed formulation, classical Monte-Carlo simulation (MCS) is also employed. To generate two-variate 2D random fields, the statistical preconditioning scheme is adopted [5]. As mentioned in [5], the discretization for random field must be sufficiently fine to reproduce the random field corresponding to white noise (i.e. when d is small). This drawback leads the analysis results to show some plateau region in coefficient of variation (COV) of response for small value of d, as shown in Figs. 1 and 4.

5.2. In-plane structures

A square in-plane structure with a constant traction in y-direction is analyzed. The material constants are: $E_o = 2.1 \times 10^6$, $\nu_o = 0.2$. The COV of displacement is found at the upper right corner.

In case of plane stress state, the correlation effects of multiple uncertain parameters are relatively small excepting in x-displacement for large correlation distance d. Figure 1 shows the COV of displacement as a function of d for plane strain state. The maximum COV in x direction is generally increased when compared with the case of sole uncertain parameter. In case of y-displacement, it depends on the value of $\rho_{\nu E}$: increase when positive and vice versa. The degree of influence is evaluated to reach about 9%. As seen in Fig. 1, the results of proposed weighted integral (WI) formulation are in good agreement with those of classical MCS. The COV of response as a function of (COV_E, COV_ν) for d = 1000.0 is given in Fig. 2, which shows degree of influence of each uncertain parameter.

5.3. Plate structures

Square plates with simple support and clamped support are analyzed. A unit distributed load is applied on the top surface in the downward direction. The variability of center-point displacement is investigated (Fig. 3). The material constants are: mean Young's modulus $E_o = 10.29 \times 10^3$, thickness $t = 1.0$ and mean Poisson ratio $\nu_o = 0.20$.

Generally, the positive correlation increases the variability and vice versa: Fig. 4. The degree of influence

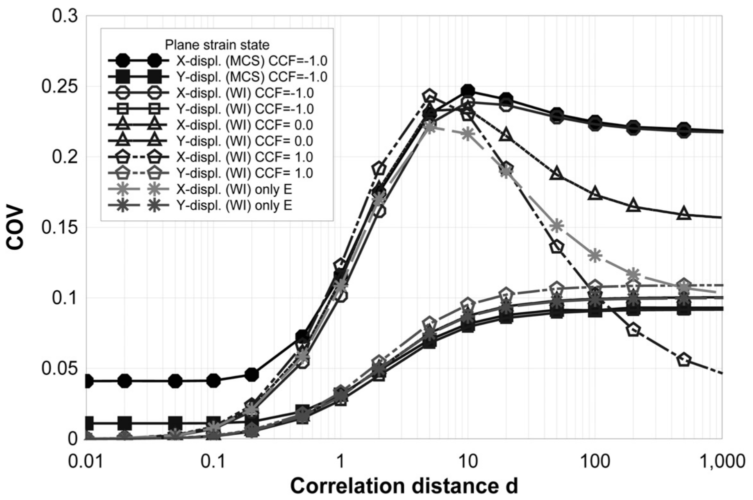
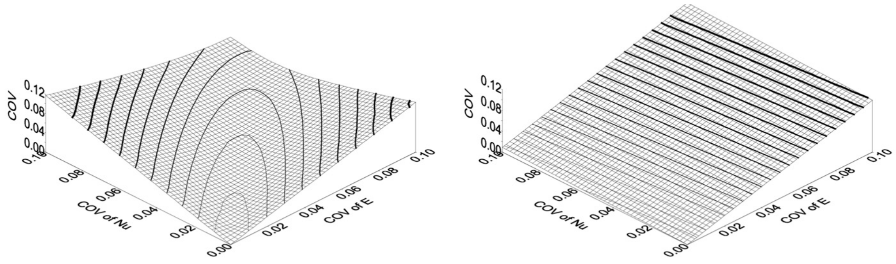


Fig. 1. COV of displacement in in-plane structure (plane strain) for no correlation (CCF = 0.0), perfect negative and positive correlation (CCF = −1.0 and 1.0).



Max=(0.0,0.1,0.118) Max=(0.1,0.1,0.109)

(a) x-displacement (b) y-displacement

Fig. 2. COV of displacement in in-plane structure for each (COV_E, COV_ν), (CCF = 1.0 and correlation distance d = 1000.0).

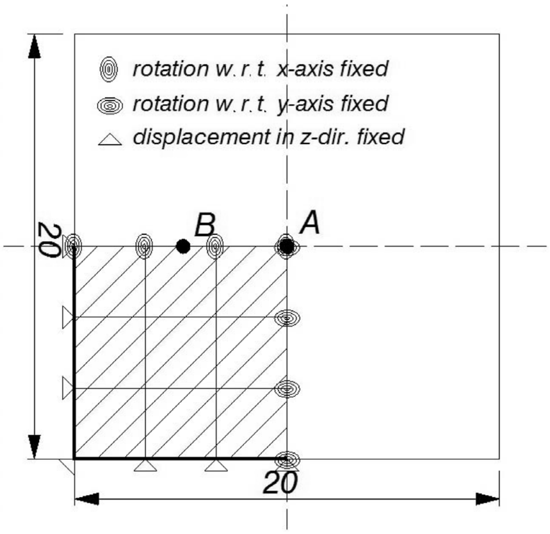


Fig. 3. Geometry of example plate: simple support.

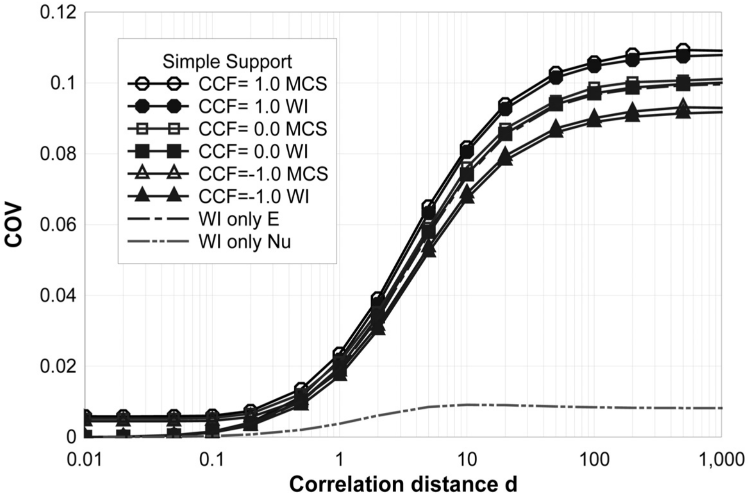


Fig. 4. COV of center displacement of simply supported plate as a function of correlation distance d: note: CCF = correlation coefficient.

is about 8% and 7% for simple and clamped support conditions. It has to be noted that these are specific values for the Poisson ratio of 0.2. In case of no correlation (CCF = 0.0), the COV of response is the same as the coefficient of variation of uncertain parameters. Though the variability obtained by the proposed formulation shows consistent under-estimation to that of classical MCS, the results of proposed formulation show good agreement with those of classical MCS.

6. Conclusions

In this study, a formulation in the stochastic finite element analysis to determine the statistical behavior of in-plane and plate structures due to the multiple uncertain material parameters is proposed. The constitutive matrix, which is a function of uncertain elastic modulus and Poisson ratio, is decomposed into infinite sub-matrices and is truncated up to 4-th terms for computational purpose. The effects of correlated uncertain parameters are examined quantitatively and qualitatively, and it is also demonstrated that the results of the proposed scheme are in good agreement with those of classical Monte-Carlo simulation.

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