A censored closure for predicting the extreme response of oscillators with non-linear damping

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Abstract

The use of the Advanced Censored Closure, recently proposed by the authors for predicting the extreme response of linear structures vibrating under random processes, is extended to oscillators with non-linear damping driven by stationary white noise. The proposed approach requires the preventive knowledge of mean upcrossing rate and spectral bandwidth of the response, which are estimated through the stochastic averaging method. A numerical application to an oscillator with a linear-plus-cubic damping is presented, and the results are compared with those of the classical Poisson approach, and of Monte Carlo simulations.

Keywords: Computational stochastic mechanics; Random vibration; Reliability analysis; First passage problem; Censored closure; Gumbel distribution

1. Introduction

The stochastic analysis of structural and mechanical systems subjected to dynamic actions of a random nature has become very popular in the last decades, since in a number of engineering situations deterministic approaches are quite unsatisfactory.

When the excitation is modelled as a Gaussian process, and the system exhibits a linear behaviour, the response is Gaussian too. In this case, then, the knowledge of mean value and standard deviation fully defines the response from a probabilistic point of view. In many cases, however, due to a non-linear behaviour of the system, the response may be significantly non-Gaussian, and higher-order statistics are required.

Unfortunately, the mere probabilistic characterization of the response is not sufficient in a reliability analysis. In fact, under the assumption that a vibrating system fails as soon as its response first exits a given safe domain, the statistics of the first passage time have to be estimated. This is recognized to be one of the most complicated problems in stochastic mechanics, and exact solutions have not been derived, even in the simplest case of SDoF linear oscillators under stationary white noise; hence, a number of approximations are available in the literature.

Among these, the most popular one is the so-called 'Poisson approach' (e.g. [1]), in which the response upcrossings of a deterministic threshold are assumed to be independent events. This classical approach, however, proved to be too conservative when the response is narrowband (e.g. because the system is lightly damped), and/or when the threshold is not high enough with respect to the standard deviation of the response. In these situations, in fact, consecutive response upcrossings are far from being independent, as they tend to occur in clumps, whose mean size depends on the spectral bandwidth of the response. The latter, then, has to be somehow accounted for in order to improve the results.

The Gaussian censored closure technique proposed by Senthilnathan and Lutes [2] reveals the same bounds, since the clumping tendency of the upcrossings is completely neglected. In order to overcome this shortcoming, Muscolino and Palmeri [3,4] recently introduced an expedient 'censorship factor', which can be related to the spectral bandwidth of the response; the use of the Gumbel model as guest probability density function (PDF) for the extreme response, instead of the Gaussian one, further improves the results. Effectiveness, accuracy and computational advantages of this formulation have been proved in the reliability analysis

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of linear structures, even in the most general case of MDoF systems subjected to coloured noises [5].

The aim of this paper is to extend the use of the latter technique, termed advanced censored closure (ACC), to SDoF oscillators featuring a non-linear damping, and subjected to a stationary white noise. To the best knowledge of the authors, this is the first time in which a censored closure is consistently applied to non-linear systems, given that in the pioneering work of Suzuki and Minai [6] the response of hysteretic oscillators is assumed to be Gaussian.

2. Proposed approach

Let us considerer the random vibration of a SDoF oscillator with a non-linear damping, driven by a zero-mean, stationary white noise W_t of power spectral density (PSD) S_0:

$$m\,\ddot{X}_t + c\big(X_t, \dot{X}_t\big)\,\dot{X}_t + k\,X_t = W_t \tag{1}$$

where X_t is the random process which describes the motion; m and k are the inertia and the elastic stiffness, respectively; $c(x,\dot{x})$ is the even function that defines the damping law; and where the over-dot denotes the derivate with respect to the time t.

Usually, the non-dimensional time $s = \omega_0 t$, with $\omega_0 = \sqrt{k/m}$, is introduced in order to pose the equation of motion in a reduced form. In so doing, one obtains:

$$X''_s + 2\,\zeta_0\,h\big(X_s, X'_s\big)\,X'_s + X_s = g\,\bar{W}_s \tag{2}$$

where the prime denotes the derivative with respect to s; $g = \sqrt{2\,\pi\,\omega_0\,S_0}\big/k$ defines the relative strength of the excitation; $\bar{W}_s$ is a non-dimensional, zero-mean, stationary white noise of unitary intensity, i.e. with autocorrelation function that is a Dirac delta function, $\mathrm{E}\langle X_r X_s\rangle = \delta(|s-r|)$, $\mathrm{E}\langle\cdot\rangle$ being the expectation operator; and where, finally, ζ_0 and $h(x,x')$ account for the linear and the non-linear portions of the damping law, respectively:

$$\zeta_0 = \frac{c(0,0)}{2\,m\,\omega_0}; \qquad h\big(X_s, X'_s\big) = \frac{c\big(X_s, \omega_0\,X'_s\big)}{c(0,0)} \tag{3}$$

2.1. Response statistics

From a probabilistic point of view, the state variables X_s and X'_s in stationary conditions are fully characterized by the knowledge of the joint PDF $p_{X,X'}(x,x')$. In a number of engineering situations, when the exact solution is not available, this function can be effectively evaluated via the stochastic averaging (SA) method [7,8], in which the motion is assumed to be pseudo-harmonic, that is:

$$X_s = A_s\,\cos(s + \Phi_s); \qquad X'_s = -A_s\,\sin(s + \Phi_s) \tag{4}$$

amplitude A_s and phase Φ_s constituting a 2-variate random process 'slowly' varying with respect to the non-dimensional time s. The method enables to estimate the effective damping ratio, in a harmonic balance sense, through the expression:

$$\zeta_{\mathrm{eff}}(a) = \frac{1}{2\pi\,a}\int_0^{2\pi} 2\,\zeta_0\,h(a\,\cos\vartheta, -a\,\sin\vartheta)\,a\,\sin^2\vartheta\,\mathrm{d}\vartheta \tag{5}$$

The knowledge of the latter quantity is sufficient to evaluate the Rayleigh-like approximate PDF of the amplitude:

$$p_A(a) = \frac{1}{N}\frac{a}{g}\exp\left[-\frac{4}{g^2}\int a\,\zeta_{\mathrm{eff}}(a)\,\mathrm{d}a\right] \tag{6}$$

where N is just a normalization constant, that is: $\int_0^{+\infty} p_A(a)\,\mathrm{d}a = 1$. Accordingly [8], the joint PDF of X_s and X'_s is given by:

$$p_{X,X'}(x,x') = \frac{1}{2\pi\sqrt{x^2 + x'^2}}\,p_A\Big(\sqrt{x^2 + x'^2}\Big) \tag{7}$$

which allows evaluating the mean upcrossing rate of a deterministic threshold b:

$$\nu_X^+(b) = \int_0^{+\infty} x'\,p_{X,X'}(b,x')\,\mathrm{d}x' \tag{8}$$

Finally, the one-sided PSD of the stationary response is given by [8]:

$$G_X(\omega) = \frac{2}{\pi\,\omega_0}\int_0^{+\infty} \frac{\zeta_{\mathrm{eff}}(a)\;a^2}{\Big[(\omega/\omega_0)^2 - 1\Big]^2 + [2\,\zeta_{\mathrm{eff}}(a)\,\omega/\omega_0]^2} \times p_A(a)\,\mathrm{d}a \tag{9}$$

which allows measuring its spectral bandwidth through the non-dimensional parameter $0 < q_X < 1$:

$$q_X = \sqrt{1 - \frac{\lambda_{1,X}^2}{\lambda_{0,X}\,\lambda_{2,X}}}; \qquad \lambda_{i,X} = \int_0^{+\infty} \omega^i\,G_X(\omega)\,\mathrm{d}\omega \tag{10}$$

2.2. Reliability analysis

Generally speaking, the reliability function $\mathcal{R}(n)$ associated with the response process X_s can be defined as the probability that the first passage time T_1 of a given safe domain does not occur prior to n cycles of the response, i.e. $\mathcal{R}(n) = \mathrm{Prob}\langle\omega_0 T_1 \geq n\rangle$. In many engineering applications, the safe domain is bounded by a so-called (double) D-barrier of level $b > 0$, so that X_s remains in the safe domain until $-b \le X_s \le b$. One can see that in this case the reliability function is identical to the evolutionary cumulative distribution function (CDF) $F_Y(b;n)$ of the extreme response process $Y_n = \max\{|X_s|, 0 \le s \le n\}$, that is:

$$\mathcal{R}(n) \equiv \mathrm{Prob}\langle Y_n \le b\rangle = F_Y(b;n) \tag{11}$$

By neglecting the spectral bandwidth of the response, the latter can be evaluated through the classical Poisson approach (e.g. [1]). When the system is assumed to start from the stationary conditions, one obtains:

$$F_Y(b;n) = F_{|X|}(b)\, \exp\left[-\frac{2\,\nu_X^+(b)\,n}{\omega_0}\right] \tag{12}$$

where $\nu_X^+(b)$ is the mean upcrossing rate of Eq.(8), and $F_{|X|}(b)$ is the CDF of the absolute value of the response, given by:

$$F_{|X|}(b) = 2\int_0^b p_X(x)\,\mathrm{d}x; \qquad p_X(x) = \int_{-\infty}^{+\infty} p_{X,X'}(x,x')\,\mathrm{d}x' \tag{13}$$

Finally, the mean value $\mu_Y(n)$ and the standard deviation $\sigma_Y(n)$ of the non-stationary process Y_n are:

$$\mu_Y(n) = \mathrm{E}\langle Y_n\rangle = \int_0^{+\infty} b\,\frac{\partial F_Y(b;n)}{\partial b}\,\mathrm{d}b;$$
$$\sigma_Y(n) = \sqrt{\mathrm{E}\left\langle [Y_n - \mu_Y(n)]^2\right\rangle} = \sqrt{\int_0^{+\infty} [b-\mu_Y(n)]^2\,\frac{\partial F_Y(b;n)}{\partial b}\,\mathrm{d}b} \tag{14}$$

As an alternative, the evolutionary second-order statistics of Y_n can be evaluated by means of the advanced censored closure (ACC) method [6], in which the differential equations governing the first two statistical moments $m_{i,Y}(n) = \mathrm{E}\langle Y_n^i\rangle$ (with $i = 1,2$), and the associated initial values, are derived in the form:

$$m'_{i,Y}(n) = 2\,i\,\chi(n)\int_0^{+\infty} \nu_X^+(b)\,b^{i-1}\,\Phi_Y(b;n)\,\mathrm{d}b\,;$$
$$m_{i,Y}(0) = 2\int_0^{+\infty} x^i\,p_X(x)\,\mathrm{d}x \tag{15}$$

where $\Phi_Y(b;n)$ is the guest CDF of the extreme response after n cycles, for which the Gumbel model is used:

$$\Phi_Y(b;n) = \exp\left\{-\exp\left[-\frac{b-\eta_Y(n)}{\kappa_Y(n)}\right]\right\} \tag{16}$$

the parameters $\eta_Y(n) = \mu_Y(n) - 0.5772\kappa_Y(n)$ and $\kappa_Y(n) = 0.7797\sigma_Y(n)$ accounting for position and spread, respectively; and where the non-dimensional censorship factor $\chi(n)$ has to be estimated through:

$$\chi(n) = \mathrm{E}\langle\beta(Y_n)\rangle = \int_0^{+\infty} \beta(b)\,\frac{\partial\left[F_{|X|}(b)\,\Phi_Y(b;n)\right]}{\partial b}\,\mathrm{d}b \tag{17}$$

$\beta(b)$ being the semi-empirical correction term proposed by Vanmarcke [9] in the reliability analysis of stationary Gaussian processes:

$$\beta(b) = \frac{1-\exp\left(-1.253\,q_X^{1.2}\,b/\sigma_X\right)}{1-\exp\left[-0.5\,(b/\sigma_X)^2\right]} \tag{18}$$

in which the parameter q_X is given by Eq. (10).

The effective step-by-step technique presented in detail in [6] can be used to numerically solve Eq. (15). Finally, the evolutionary mean value and the standard deviation of Y_n are:

$$\mu_Y(n) = m_{1,Y}(n); \qquad \sigma_Y(n) = \sqrt{m_{2,Y}(n) - \mu_Y(n)^2} \tag{19}$$

3. Numerical application

The approximate formulations dealt with in the previous section, namely the classical Poisson approach and the proposed ACC technique, are applied to a SDoF oscillator with a linear-plus-cubic damping.

The damping law in Eq. (2) is defined by $\zeta_0 = 0.003$ and $h(x,x') = 1 + 0.15x'^2$; two strengths of excitation are selected: $g = 1$, and $g = 6$.

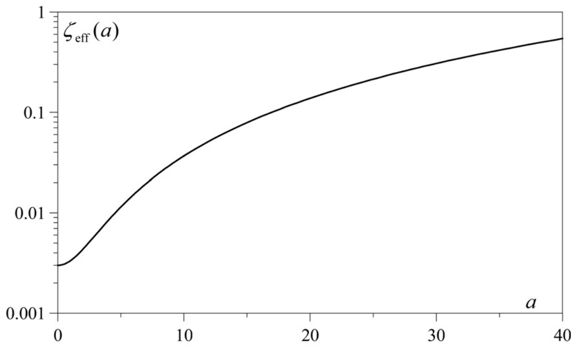

Fig. 1. Effective damping ratio.

In a first stage, the SA method is used to evaluate the approximate statistics of the stationary response. In Fig. 1 the effective damping ratio, $\zeta_{\mathrm{eff}}(a) = 0.003 + 0.0003375a^2$, is depicted: it is worth noting that this is a

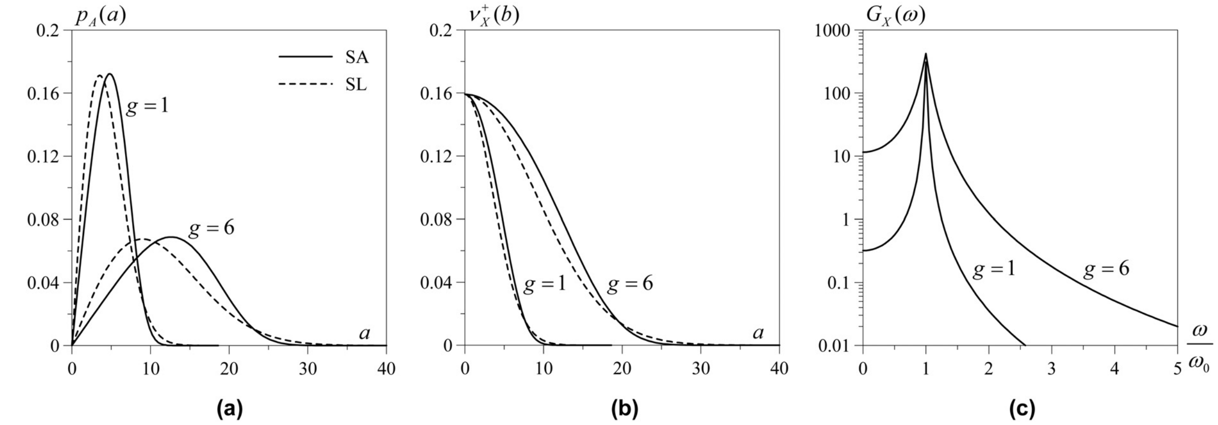

Fig. 2. PDF of the amplitude (a); mean upcrossing rate (b), and PSD of the response (c).

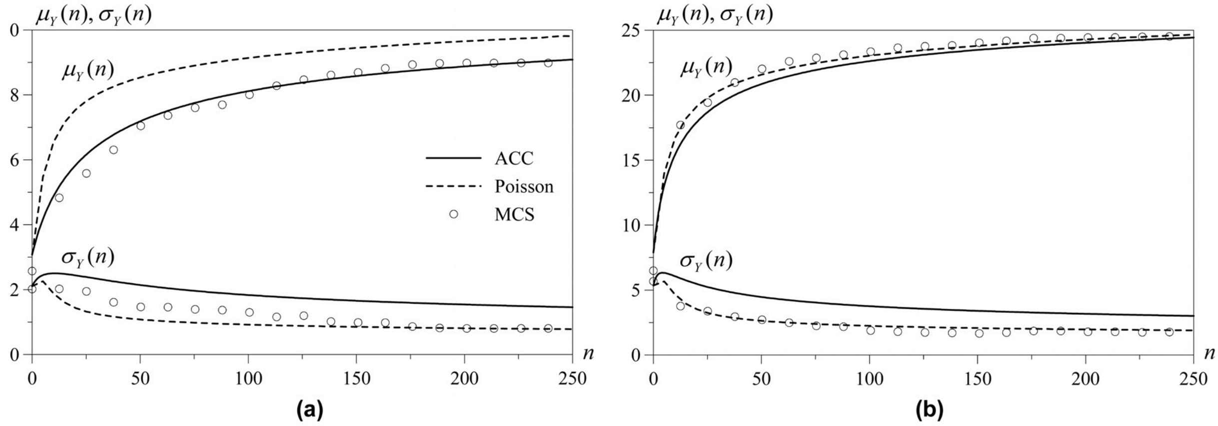

Fig. 3. Evolutionary mean value and standard deviation of the extreme response: (a) narrowband response; (b) broadband response.

monotonic increasing function of the amplitude only. In Fig. 2(a) and Fig. 2(b), respectively, the PDF of the amplitude (Eq. (6)) and the mean upcrossing rate of the response (Eq. (8)) given by the SA method are compared with those given by the well-known stochastic linearization (SL) method [10], which works under the assumption that the response process is Gaussian. One can see that, even for the lower value of g, the discrepancy is not negligible; as a consequence, the deviation from the Gaussianity needs to be taken into account in the reliability analysis. In Fig. 2(c) the PSD of the responses (Eq. (9)) are shown. It is worth noting that the higher excitation not only raises the energy content over all the frequencies, but also increases the spectral bandwidth: i.e. for $g = 1$ the response process is narrowband, while for $g = 6$ it becomes broadband.

In a second stage, the evolutionary mean value and standard deviation of the extreme response are evaluated. Figure 3 tells that, independently of the spectral bandwidth of the response, the mean value of the proposed ACC (solid lines) is in good agreement with the result of 50 Monte Carlo simulations (MCS) circles, while the standard deviation is overestimated. This is mainly due to the discrepancy between the right-hand tail of the Gumbel distribution used in the analysis, and that of the actual distribution of the extreme response. On the contrary, the classical Poisson approach (dashed lines) proves to be accurate in the case of the higher excitation only (Fig. 3(b)), i.e. when the response process is broadband, while it is too conservative in the other case (Fig. 3(a)).

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