Large strain anisotropic plasticity including effects of plastic spin

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Abstract

A constitutive theory for orthotropic materials is briefly summarized. The constitutive theory allows for anisotropy of the elastic response and anisotropy resulting from both the yield function and the kinematic hardening. The formulation also includes a constitutive theory, thermodynamically consistent, for the plastic spin and the re-orientation of the anisotropy properties towards a more favorable position from a thermodynamic point of view. The model is formulated in terms of the Lee decomposition and Hencky strain measures. A numerical example to illustrate the model is presented.

Keywords: Anisotropic plasticity; Large strains; Plastic spin; Logarithmic strains; Lattice rotation

1. Introduction

During the recent years, large strain plasticity received increasing attention. The use of hyperelastic relations rendered a thermodynamically consistent framework, in which elastic procedures (or elastic strains in plastic processes) do not dissipate energy (see [1] for an earlier contribution and a discussion in [2]). The use of the multiplicative (or Lee) decomposition [3] has also been a major milestone because it is based on the micromechanics of crystals and has an immediate continuum interpretation [4]. Finally, the combination of a hyperelastic stored energy function based on Hencky strain measures, physically motivated [5], and the use of the exponential integration algorithm yielded a simple extension of the small strains algorithms to the large strains framework, both for isotropic materials [6] and for anisotropic behavior where the anisotropy is due to the kinematic hardening [4, 7]. As mentioned in [4], these algorithms may be used with arbitrary anisotropic yield functions. Nonetheless, that formulation is restricted to elastic isotropy.

Effective algorithms for anisotropic plasticity are difficult to reach due to the inherent difficulties introduced by anisotropy. Some publications are available on computational anisotropic plasticity, see for example [8–10], but these algorithms are not developed following the successful framework using the Lee decomposition, logarithmic strains and exponential mapping.

On the other hand, recent experiments conducted on anisotropic specimens [11] have shown that, for moderate strains (to about 5%), the preferred directions of orthotropy rotated to more favorable directions given by the new stress–strain state, whereas the shape of the yield surface remained basically unaltered.

In the present work, we briefly summarize a constitutive theory for anisotropic plasticity, thermodynamically consistent, which employs logarithmic strain measures, the multiplicative Lee decomposition, the exponential mapping and the rotation of the directions of the anisotropy properties. The details of the formulation are given in [12], see also [4]. In the next sections we will follow the notation of [13].

2. Continuum formulation

The theory is based on the multiplicative Lee decomposition of the deformation gradient $\mathbf{X}$ into an elastic $\mathbf{X}^E$ and a plastic part $\mathbf{X}^p$, i.e. $\mathbf{X} = \mathbf{X}^E\mathbf{X}^p$. The spatial velocity gradient $\mathbf{l}$ is decomposed as:

$$\mathbf{l} = \mathbf{l}^E + \mathbf{l}^p = \dot{\mathbf{X}}^E(\mathbf{X}^E)^{-1} + \mathbf{X}^E\left[\dot{\mathbf{X}}^p(\mathbf{X}^p)^{-1}\right](\mathbf{X}^E)^{-1} \quad (1)$$

The tensor $\mathbf{L}^p := \dot{\mathbf{X}}^p(\mathbf{X}^p)^{-1}$ is the modified plastic velocity gradient. The pull-back of $\mathbf{l}$ to the intermediate configuration given by $\mathbf{X}^E$ yields $\mathbf{L} = \mathbf{L}^E + \mathbf{C}^E\mathbf{L}^p$, where

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$\mathbf{C}^E = \mathbf{X}^{E\,T}\mathbf{X}^E$ is the right Cauchy–Green elastic deformation tensor. The elastic deformation rate tensor is $\mathbf{D}^E = sym\ (\mathbf{L}^E)$ and the modified plastic deformation tensor is $\mathbf{D}^p = sym\ (\mathbf{L}^p)$. The modified plastic spin is $\mathbf{W}^p = skw\ (\mathbf{L}^p)$. The logarithmic strain tensor is defined as $\mathbf{E}^E = \frac{1}{2} \ln \mathbf{C}^E$. Of course, once a strain measure is known any other strain measure may be obtained by the proper fourth-order mapping tensors, and the same holds for strain rate tensors [4,12,13]. Then, stress measures may also be related by fourth-order tensors based on work-conjugacy.

Let us define the Mandel stress tensor $\Xi = \mathbf{C}^E\mathbf{S}$, where $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor. The symmetric and skew parts of the Mandel stress tensor are obtained by the expressions:

$$\Xi_s = \mathbf{T} : \mathbf{S}^M \quad \text{and} \quad \Xi_w = \mathbf{T} : \mathbf{W}^M = \mathbf{E}^E\mathbf{T} - \mathbf{T}\mathbf{E}^E \tag{2}$$

where $\mathbf{T} := \partial\psi/\partial\mathbf{E}^E$ is the Kirchhoff stress tensor, ψ is the free energy function and $\mathbf{S}^M$ and $\mathbf{W}^M$ are fourth-order mapping tensors, functions of the elastic strains (see details in [12]). The free energy function is split into an elastic part, $\mathcal{W}$, and a plastic (with plastic hardening) part, $\mathcal{H}$, i.e. $\psi = \mathcal{W} + \mathcal{H}$. The elastic part is assumed to be anisotropic and may be expressed in terms of the logarithmic strains as:

$${}^t\mathcal{W} = U({}^tJ) + \mu\ {}^t\mathbf{E}^E : {}^t\mathbf{A}\}\}: {}^t\mathbf{E}^E \tag{3}$$

where ${}^t\mathbf{A}$ is the elastic anisotropy tensor, whose preferred directions may rotate at a speed given by $\mathbf{W}^A$. In this expression $U\ ({}^tJ)$ is the volumetric component, $J = \det\ (\mathbf{X}^E)$ and μ plays the role of a shear modulus.

On the other hand, the tensor $\mathbf{L}^p$ can be split into a symmetric part $\mathbf{D}^p$ (the plastic deformation rate tensor) and a skew part $\mathbf{W}^p$ (the modified plastic spin). It can be argued (see [4,12]) that the tensor:

$${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w := \exp\left({}^{t+\Delta t}\mathbf{W}^p \Delta t\right) \tag{4}$$

is a measure of the incremental plastic rotation due to lattice dislocations. The tensor ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$ defines, from the stress-free configuration, a configuration in which the plastic rotations are frozen during plastic flow. This configuration is specially suitable for performing the stress integration and for constructing the continuum formulations. We label objects in this configuration by an underlining arrow, as in $\underleftarrow{\mathbf{E}}^E$, i.e. ${}^{t+\Delta t}\underleftarrow{\mathbf{E}}^E = {}^{t+\Delta t}_{\ \ t}\mathbf{R}^{wT}\ {}^{t+\Delta t}\mathbf{E}^E\ {}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$. Then the rate of the stored energy function $\dot{\mathcal{W}}$ is:

$$\dot{\mathcal{W}} = \mathbf{T} : \underleftarrow{\mathcal{L}}\mathbf{E}^E + \mathbf{T}_w : \mathbf{W}^A \tag{5}$$

where $\underleftarrow{\mathcal{L}}\ (\cdot)$ is a Lie derivative with ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$ acting as gradient and $\mathbf{T}_w := \mathbf{E}^E\ \mathbf{T} - \mathbf{T}\ \mathbf{E}^E \equiv \Xi_w$. Similar expressions apply for the hardening function, $\mathcal{H}$, and the rate of hardening, $\dot{\mathcal{H}}$.

All these equations, inserted in the dissipation equation, yield the following expressions after some considerations and manipulations:

$$(-\mathbf{T} + \partial\psi/\partial\mathbf{E}^E) : \underleftarrow{\mathcal{L}}\mathbf{E}^E = 0 \tag{6}$$

$$\dot{\mathcal{D}}^p := \Xi_s : \mathbf{D}^p + \Xi_w : \mathbf{W}^p - \mathbf{T}_w : \mathbf{W}^A - \mathbf{B}_s : \underleftarrow{\mathcal{L}}\mathbf{E}^i - \mathbf{B}_w : \mathbf{W}^H - \kappa\dot{\zeta} - \kappa_w\dot{\xi} \geq 0 \tag{7}$$

where $\mathbf{B}_s$ is the backstress tensor, $\mathbf{B}_w := \mathbf{E}^i\ \mathbf{B}_s - \mathbf{B}_s\ \mathbf{E}^i$; and $\mathbf{E}^i$ are the tensorial strain-like internal variables. The scalars κ and κ_w are the effective stress-like internal variables (current yield stress and yield couple-stress). The scalars ζ and ξ are the effective strain-like internal variables (effective plastic strain and effective plastic rotation).

We *assume* now – to keep the formulation focused, but without loss of generality – that the elastic region is enclosed by two yield functions $f_s\ (\Xi_s, \mathbf{B}_s, \kappa)$ and $f_w\ (\Xi_w, \mathbf{B}_w, \kappa_w)$, and the Lagrangian for the constrained problem is $L = \dot{\mathcal{D}}^p - \dot{\iota} f_s - \dot{\gamma} f_w$, where $\dot{\iota}$ and $\dot{\gamma}$ are the consistency parameters. If we claim that the principle of maximum dissipation holds, the stress and other internal variables are such that $\nabla L = 0$, i.e. for the yield function expressions given:

$$\nabla L = 0 \Rightarrow \begin{cases} \dfrac{\partial L}{\partial \Xi_s} = 0 \Rightarrow \mathbf{D}^p = \dot{\iota}\dfrac{\partial f_s}{\partial \Xi_s} \\[2ex] \dfrac{\partial L}{\partial \Xi_w} = 0 \Rightarrow \mathbf{W}^d = \dot{\gamma}\dfrac{\partial f_w}{\partial \Xi_w} \\[2ex] \dfrac{\partial L}{\partial \mathbf{B}_s} = 0 \Rightarrow \underleftarrow{\mathcal{L}}\ \mathbf{E}^i = -\dot{\iota}\dfrac{\partial f_s}{\partial \mathbf{B}_s} \\[2ex] \dfrac{\partial L}{\partial \mathbf{B}_w} = 0 \Rightarrow \mathbf{W}^H = -\dot{\gamma}\dfrac{\partial f_w}{\partial \mathbf{B}_w} \\[2ex] \dfrac{\partial L}{\partial \kappa} = 0 \Rightarrow \dot{\zeta} = -\dot{\iota}\dfrac{\partial f_s}{\partial \kappa} \\[2ex] \dfrac{\partial L}{\partial \kappa_w} = 0 \Rightarrow \dot{\xi} = -\dot{\gamma}\dfrac{\partial f_w}{\partial \kappa_w} \end{cases} \tag{8}$$

where $\mathbf{W}^d := \mathbf{W}^p - \mathbf{W}^A$ and $\mathbf{W}^H$ is the spin of the hardening anisotropy tensor. These expressions are the associated flow and hardening rules for general elasto-plasticity at finite strains. If, as usual, the enclosure of the elastic region is expressed in the form of $f_s\ (\Xi_s - \mathbf{B}_s \ldots)$, $f_w\ (\Xi_w - \mathbf{B}_w \ldots)$, then for associated plasticity the following relationships are automatically enforced:

$$\underleftarrow{\mathcal{L}}\mathbf{E}^i = \mathbf{D}^p \text{ and } \mathbf{W}^H = \mathbf{W}^d \tag{9}$$

3. Small strain formulation

In order to isolate the effect of large strains from the anisotropy rotations, we developed a small strain algorithm, in which higher order terms are neglected except for those directly related to the rotation of the orthotropy directions. These rotation terms must be kept in order to capture such phenomena with a small strain formulation. Hence, this small strain formulation is in some sense not a rigorous limit of the large strain formulation. However, it is a convenient formulation to study the effect of the rotation of the anisotropy directions.

The small strain formulation is obtained using the limits:

$$\Xi_s = \boldsymbol{\sigma}; \quad \Xi_w = \boldsymbol{\sigma}_w := \boldsymbol{\varepsilon}^e \boldsymbol{\sigma} - \boldsymbol{\sigma} \boldsymbol{\varepsilon}^e \tag{10}$$

where $\boldsymbol{\sigma}$ are the Cauchy stresses and $\boldsymbol{\varepsilon}^e$ are the small elastic strains. Note that usually the couple-stresses $\boldsymbol{\sigma}_w$ are two orders of magnitude smaller than $\boldsymbol{\sigma}$ if small strains are strictly enforced. Also, we use the limits:

$$\underset{\leftarrow}{\mathcal{L}}\mathbf{E}^e = \dot{\varepsilon}^e; \quad \mathbf{D}^p = \dot{\varepsilon}^p; \quad \mathbf{W}^p = \dot{\boldsymbol{\omega}}^p \tag{11}$$

where $\dot{\varepsilon}^p$ are the plastic strain rates and $\dot{\boldsymbol{\omega}}^p$ are the plastic spins. The incremental plastic rotation is approximated by ${}^{t+\Delta t}_{t}\mathbf{r}^w = \mathbf{I} + \Delta t^{t+\Delta t}\dot{\boldsymbol{\omega}}^p$, and the plastic dissipation equation is:

$$\dot{\mathcal{D}}^p = \boldsymbol{\sigma} : \dot{\varepsilon}^p + \boldsymbol{\sigma}_w : \dot{\boldsymbol{\omega}}^d - \boldsymbol{\beta}_s : \dot{\varepsilon}^i - \boldsymbol{\beta}_w : \dot{\boldsymbol{\omega}}^H - \kappa\dot{\zeta} - \kappa_w\dot{\xi} \geq 0 \tag{12}$$

where $\boldsymbol{\beta}_s$ are the back-stresses, $\boldsymbol{\beta}_w$ are the back-couple-stresses, $\dot{\boldsymbol{\omega}}^d := \dot{\boldsymbol{\omega}}^p - \dot{\boldsymbol{\omega}}^A$. The tensors $\dot{\boldsymbol{\omega}}^A$ and $\dot{\boldsymbol{\omega}}^H$ are the spins of the elastic anisotropy tensor $\mathbb{A}$ and the kinematic hardening anisotropy tensor $\mathbb{H}$. In this way, all above large strain expressions are reduced to the small strain case.

4. Yield function

In order to use explicit expressions, we employ a Hill-type yield function expression for the symmetric part (see [14]), which in terms of Cauchy stresses is:

$$f_s = \tfrac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\beta}_s) : \mathbb{A}_s^p : (\boldsymbol{\sigma} - \boldsymbol{\beta}_s) - \tfrac{1}{3}\kappa^2 \tag{13}$$

and for the skew part:

$$f_w = \tfrac{1}{2}(\boldsymbol{\sigma}_w - \boldsymbol{\beta}_w) : \mathbb{A}_w^p : (\boldsymbol{\sigma}_w - \boldsymbol{\beta}_w) - \tfrac{1}{3}\kappa_w^2 \tag{14}$$

where the effective plastic rotation rate ξ' is obtained from the effective plastic strain rate $\dot{\zeta}$ as $\xi' = \dot{\zeta}\,\langle f_w \rangle / \eta$, where $\langle \cdot \rangle$ defines the ramp function and η is a material parameter. The tensors $\mathbb{A}_s^p$ and $\mathbb{A}_w^p$ are anisotropy tensors whose preferred directions rotate.

With the above theory, a fully implicit integration algorithm can be established [12].

5. Example

In this example, we illustrate, to a limited extent, the behavior of the model. We consider a single plane strain square element. The isochoric prescribed strains in the Cartesian reference system are:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & -\varepsilon_x & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathrm{X}} \tag{15}$$

The third anisotropy preferred direction is given by $\mathbf{P}_3 = [0, 0, 1]^T_{\mathrm{X}}$. For simplicity, the symmetric

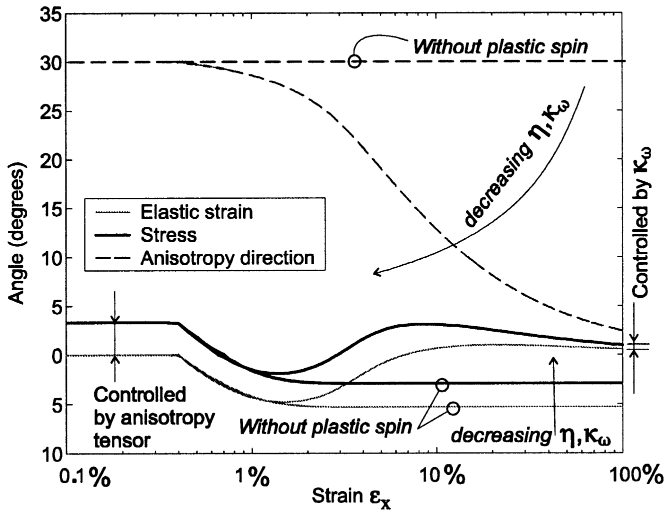

Fig. 1. Rotation of the principal directions of elastic strain, stress and of the anisotropy tensor with initial value ${}^0\mathbf{P}_1 = [\cos 30, \sin 30, 0]^T_{\mathrm{X}}$.

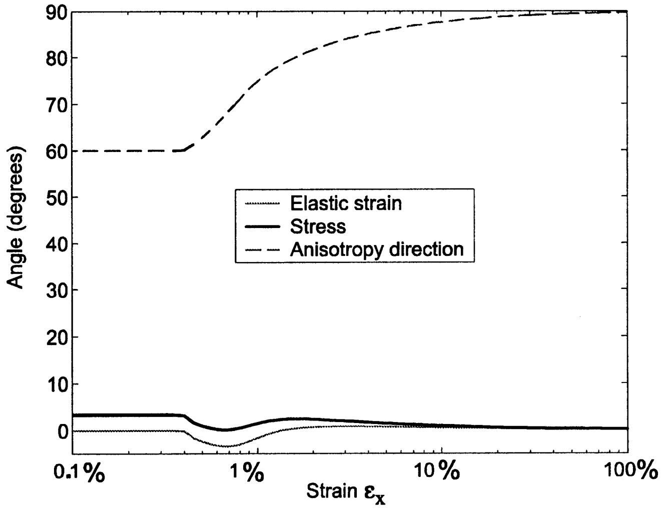

Fig. 2. Rotation of the principal directions of elastic strain, stress and of the anisotropy tensor with initial value ${}^0\mathbf{P}_1$ = [cos 60, sin 60, 0] ^T_X.

anisotropy tensors are set equal to each other and are given in Voigt notation in the principal anisotropy directions:

$$\mathbb{A} = \mathbb{A}_s^p = \begin{bmatrix} \frac{1}{3}(a_1 + a_2) & -\frac{1}{3}a_1 & -\frac{1}{3}a_2 & 0 & 0 & 0 \\ -\frac{1}{3}a_1 & \frac{1}{3}(a_1 + a_3) & -\frac{1}{3}a_3 & 0 & 0 & 0 \\ -\frac{1}{3}a_2 & -\frac{1}{3}a_3 & \frac{1}{3}(a_2 + a_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix} \quad (16)$$

where $a_1 = 0.8$, $a_2 = 1.2$, $a_3 = 1.0$, $a_4 = 0.7$, $a_5 = 1.0$, $a_6 = 1.3$ are dimensionless constants. We also use $\mathbb{A}_w^p = \mathbb{I}$, $\mu = 76.9$ MPa, $\kappa = 0.612$MPa, $H = 0$. The resulting behaviors at the stress point are shown in Figs. 1 and 2. In the numerical experiments we consider large strain values using the small strain plasticity theory, merely to illustrate the observed effects. The possible results that would be obtained using different admissible couple-stress κ_w and 'viscosity' parameters η are indicated by arrows in Fig. 1, and the particular case drawn corresponds to the values $\kappa_w = 10^{-8}$ MPa and $\eta = 10^{-9}$ MPa^2 and an initial preferred direction ${}^0\mathbf{P}_1$ = [cos 30, sin 30, 0] ^T_X. Figure 2 shows the results for ${}^0\mathbf{P}_1$ = [cos 60, sin 60, 0] ^T_X and $\kappa_w = 10^{-8}$ MPa and $\eta = 10^{-8}$ MPa^2. Both figures show that, as the strain increases, the anisotropy directions rotate so that the principal stress directions become aligned with the elastic strain principal directions to a final value governed by κ_w. This effect is controlled by the couple-stress $\boldsymbol{\sigma}_w$. The results are qualitatively similar to those obtained experimentally by Kim and Yin [11].

6. Conclusions

In this paper a thermodynamically consistent constitutive theory for anisotropic plasticity including a constitutive relation for the plastic spin is summarized. The theory is based on an anisotropic hyperelastic relation for the Kirchhoff stresses and considers both isotropic and anisotropic kinematic hardening. Constitutive relations are obtained from the yield function expressions and the principle of maximum dissipation. A small strain limit theory, preserving the plastic rotation effect is also developed and some results on the behavior of the model are shown.

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