Stochastic generalized differential quadrature formulation

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Abstract

The assumption that structures have deterministic material properties is implicitly involved in the most calculation of standard structural analysis. Material and geometric properties are assumed to constitute homogenous, one-dimensional stochastic fields, which means that the response deflection is also a stochastic field. The stochastic generalized differential quadrature method is introduced and formulated for structural analysis problems. The concept of the variability response function is extended to the stochastic differential quadrature method and used to compute spectral-distribution-free upper bounds of the response variability. In addition, the generalized differential quadrature procedure is described for a beam bending problem.

Keywords: Stochastic generalized differential quadrature; Random material and geometric properties; Response variability; Spectral-distribution-free upper bounds; Variability response function

1. Introduction

The assumption that structures have deterministic geometrical and material properties is implicitly involved in the most calculation of standard finite element structural analysis. The material and geometric properties of real structures have uncertainties, which have to be considered in structural analysis. The uncertainties of the structures are then for practical structural analysis considered through the increase of the safety factors.

The input quantities (material and geometrical properties) are assumed to constitute homogenous, stochastic fields. It means that the response deflection is also a stochastic field. The stochastic generalized differential quadrature formulation is introduced and used to find the influence of the randomness of the input quantities on the randomness of the response deflection. The first and second moments of stochastic properties are taken for describing the randomness of input quantities. The stochastic stiffness matrix is represented as a linear combination of deterministic and stochastic parts. The response variability is calculated using the first order Taylor-expansion approximation of the variability response function. The concept of variability response function, introduced by Deodatis and Shinozuka [1], is extended to GDQ and used to compute spectral-distribution-free upper bounds of response variability. The spectral-distribution-free upper bounds are very important for engineers because only mean values and the coefficient of variations are usually known about the randomness of structural properties. A numerical procedure is given and described for a beam bending problem. A few examples for beam bending are completely calculated. A very significant example is where the coefficient of variation of response deflection can become larger than the coefficient of variation of input quantities.

The aim of this paper is to formulate the generalized differential quadrature (GDQ) method for stochastic structural analysis and to extend the concept of the variability response function to stochastic GDQ.

2. Variability of input quantities

We consider a structure with spatially varying material and/or geometrical properties. Some structural property, $G(x)$, is assumed to constitute a homogenous, one-dimensional random field of the following form:

$$G(x) = G_0[1 + g(x)] \quad (1)$$

where G_0 is the expectation of this property, and $g(x)$ is a homogenous, one-dimensional, zero-mean, random

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Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

field. This random field $g(x)$ can be represented with its variance σ_{gg}^2 and autocorrelation function:

$$R_{gg}(\xi) = E[g(x+\xi)g(x)] \tag{2}$$

which leads to the variance and coefficient of variation of property $G(x)$:

$$\mathrm{Var}[\mathrm{G}(x)] = G_0\sigma_{gg}^2;\ \mathrm{COV}[G(x)] = \sigma_{gg} \tag{3}$$

3. Generalized differential quadrature method

The differential quadrature approximation of the nth derivative of the function $w(x)$ at the ith discrete point on a grid is given by a weighted linear sum of the function values at all discrete points (N points) along that direction (direction x) as

$$\frac{d^n w(x)}{dx^n} = \sum_{j=1}^{N} c_{ij}^{(n)} w(x_j) \tag{4}$$

where $c_{ij}^{(n)}$ are weighting coefficients for the nth derivative and $w(x_j)$ are function values at grid points x_j, $j = 1, 2, \ldots N$. The goal of the generalized differential quadrature method is to find a simple algebraic expression for calculating the weighting coefficients $c_{ij}^{(n)}$ for an arbitrary choice of grid points. The weighting coefficients for the mth-order derivative are given by recurrence relations in general form as, Du et al. [2],

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \quad \text{for} \quad i \neq j, \quad i,j = 1,2, \ldots N \tag{5}$$

$$c_{ij}^{(m)} = m\left(c_{ii}^{(m-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right) \quad \text{for} \quad i \neq j, \quad m = 2,3, \ldots N-1, \quad i,j = 1, \ldots N \tag{6}$$

$$c_{ii}^{(m)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(m)} \quad \text{for} \quad m = 1,2, \ldots N-1, \quad i = 1,2, \ldots N \tag{7}$$

where

$$M(x_i) = \prod_{j=1}^{N}(x - x_j) \tag{8}$$

These recurrence expressions are very useful for implementation in programming. There is no need to solve a set of algebraic equations to find the weighting coefficients, and we have no restriction on the chosen grid points. There is no influence of loading or boundary conditions on the weighting coefficients which, once calculated, can be used for the next loading or boundary conditions on the same beam. Only the new right side (the force vector) for a set of algebraic equations should be evaluated. Before further analysis the new boundary conditions should also be applied. According to these features less computational effort is required for solving any of structural problems by using generalized differential quadrature method. The application of the GDQ method for static structural problems leads, in the general case, to a set of N algebraic equation with N unknown function values at the grid points.

4. Stochastic generalized differential quadrature formulation

The standard deterministic differential quadrature formulation of any structural problem for further numerical analysis is expressed as

$$\mathbf{K}_0\mathbf{w} = \mathbf{q} \tag{9}$$

Involving the randomness of material and/or geometric properties, the formulation for the stochastic analysis is then, according to Shinozuka and Deodatis [3], given as

$$(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{w} = \mathbf{q} \tag{10}$$

where $\mathbf{K}_0$ is the deterministic stiffness matrix and $\Delta\mathbf{K}$ is the stochastic part of the stiffness matrix. The stochastic response vector is approximated with:

$$\begin{aligned} \mathrm{w} &= [\mathbf{I} - \mathbf{Q} + \mathbf{Q}^2 - \cdots]\mathbf{K}_0^{-1}\mathbf{q} \\ &= (\mathbf{I} - \mathbf{Q})\mathbf{w}_0 \end{aligned} \tag{11}$$

with the assumption that variance is sufficiently small and that

$$\mathbf{Q} = \mathbf{K}_0^{-1}\Delta\mathbf{K}$$

It now follows that the expression for expectation of response deflection is

$$\mathrm{E}[\mathbf{w}] = \mathbf{w}_0 \tag{13}$$

and the covariance matrix of the response deflection is given by:

$$\begin{aligned} \mathrm{Cov}[\mathbf{w}, \mathbf{w}] &= \mathrm{E}[(\mathbf{w} - \mathbf{w}_0)(\mathbf{w} - \mathbf{w}_0)^{\mathrm{T}}] \\ &= \mathrm{E}[\mathbf{Q}\mathbf{W}_0\mathbf{Q}^{\mathrm{T}}] \end{aligned} \tag{14}$$

where $\mathbf{W}_0 = \mathbf{w}_0\mathbf{w}_0^{\mathrm{T}}$.

According to the definition of the randomness of the structural property $G(x)$, the stochastic value of that property at any grid point x_i is then

$$G(x_i) = G_0(1 + g(x_i)) \tag{15}$$

and the stochastic value of the response deflection:

$$w(x_i) = w_0(1 + \Delta_w(x_i)) \tag{16}$$

The stochastic part of the stiffness matrix ΔK is represented as a linear combination of N random variables $g(x_i)$:

$$\Delta\mathbf{K}^{(e)} = \sum_{i=1}^{N} g(x_i)\Delta\mathbf{K}_i \tag{17}$$

The coefficient matrices $\Delta\mathbf{K}_i$, ($i = 1, \dots N$), are all deterministic. It is possible to perform a first-order Taylor expansion of $\mathbf{w}$ around the mean values of these N random variables, $g(x_i)$, where all these mean values are equal to zero:

$$\begin{aligned}
\mathbf{w} &= \mathbf{w}_0 + \sum_{i=1}^{N} g(x_i)\left[\frac{\partial \mathbf{w}}{\partial g(x_i)}\right]_{\mathrm{E}} \\
&= \left|\begin{array}{c}
\mathbf{K}\mathbf{w} \;=\; \mathbf{q} \\
\frac{\partial}{\partial g(x_i)}[\mathbf{K}\mathbf{w}]_{\mathrm{E}} = \left[\frac{\partial \mathbf{q}}{\partial g(x_i)}\right]_{\mathrm{E}} \\
\left[\frac{\partial \mathbf{K}}{\partial g(x_i)}\right]_{\mathrm{E}}\mathbf{w}_0 + \mathbf{K}_0\left[\frac{\partial \mathbf{w}}{\partial s(x_i)}\right]_{\mathrm{E}} = 0 \\
\left[\frac{\partial \mathbf{w}}{\partial g(x_i)}\right]_{\mathrm{E}} = -\mathbf{K}_0^{-1}\left[\frac{\partial \mathbf{K}}{\partial g(x_i)}\right]_{\mathrm{E}}\mathbf{w}_0
\end{array}\right| \\
&= \mathbf{w}_0 - \sum_{i=1}^{N}\mathbf{K}_0^{-1}\left[\frac{\partial \mathbf{K}}{\partial s(x_i)}\right]_{\mathrm{E}}\mathbf{w}_0 g(x_i) \\
&= \mathbf{w}_0 - \sum_{i=1}^{N}\mathbf{K}_0^{-1}\Delta\mathbf{K}_i\mathbf{w}_0 g(x_i)
\end{aligned} \tag{18}$$

where $[\partial\mathbf{w}/\partial g(x_i)]_{\mathrm{E}}$ and $[\partial\mathbf{K}/\partial g(x_i)]_{\mathrm{E}}$ are values calculated at the mean values of the random variables $g(x_i)$ and N is the number of grid points. This leads to a first-order approximation of the covariance matrix of the response vector $\mathbf{w}$:

$$\begin{aligned}
\mathrm{Cov}[\mathbf{w},\mathbf{w}] \;&=\; \mathrm{E}[(\mathbf{w}-\mathbf{w}_0)(\mathbf{w}-\mathbf{w}_0)^{\mathrm{T}}] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{K}_0^{-1}\Delta\mathbf{K}_i\mathbf{W}_0(\Delta\mathbf{K}_j)^{\mathrm{T}}(\mathbf{K}_0^{-1})^{\mathrm{T}}\mathrm{E}[g(x_i)g(x_j)]
\end{aligned} \tag{19}$$

The only unknown quantity is now the expectation $\mathrm{E}[g(x_i)g(x_j)]$. This expectation can be written using the definition of the autocorrelation function and the Wiener-Khintchine transformation [4] as

$$\begin{aligned}
\mathrm{E}[(g(x_i)g(x_j)] \;&=\; R_{gg}(x_i - x_j) \\
&= \int_{-\infty}^{\infty} S_{gg}(\kappa)e^{i\kappa(x_i-x_j)}d\kappa
\end{aligned} \tag{20}$$

where $S_{gg}(\kappa)$ is the power-spectral-density function of the stochastic field $g(x)$. Substituting this expression into Eq. (19), the covariance matrix can be obtained as:

$$\begin{aligned}
\mathrm{Cov}[\mathbf{w},\mathbf{w}] = \int_{-\infty}^{\infty} S_{gg}(\kappa)\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{K}_0^{-1}\Delta\mathbf{K}_i\mathbf{W}_0(\Delta\mathbf{K}_j)^{\mathrm{T}} \\
(\mathbf{K}_0^{-1})^{\mathrm{T}}e^{i\kappa(x_i-x_j)}d\kappa
\end{aligned} \tag{21}$$

Then, the variance vector of the response vector $\mathbf{w}$, consisting of the diagonal elements of the covariance matrix of $\mathbf{w}$, is found to be:

$$\begin{aligned}
\mathrm{Var}[\mathbf{w}] &= \int_{-\infty}^{\infty} S_{gg}(\kappa)\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{diag}(\mathbf{K}_0^{-1}\Delta\mathbf{K}_i\mathbf{w}_0)\mathbf{K}_0^{-1}\Delta\mathbf{K}_j\mathbf{w}_0 e^{i\kappa(x_i-x_j)}d\kappa \\
&= \int_{-\infty}^{\infty} S_{gg}(\kappa)\mathrm{V}(\kappa)d\kappa
\end{aligned} \tag{22}$$

where $\mathbf{V}(\kappa)$ is the first-order approximation of the variability response function [1], defined as

$$\begin{aligned}
\mathrm{V}(\kappa) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{diag}(\mathbf{K}_0^{-1}\Delta\mathbf{K}_i\mathbf{w}_0)\mathbf{K}_0^{-1}\Delta\mathbf{K}_j\mathbf{w}_0 \\
[\cos(\kappa x_i)\cos(\kappa x_j) + \sin(\kappa x_i)\sin(\kappa x_j)]
\end{aligned} \tag{23}$$

and diag[·] means a diagonal matrix whose diagonal components are equal to the vector within the parentheses.

If we consider a specific degree of freedom w_i, and its corresponding response variability component $V_i(\kappa)$, of the coefficient of variation is found to be bounded as:

$$\mathrm{COV}[w_i] \le \sigma_{gg}\frac{\sqrt{V_i(\kappa^*)}}{\|\mathrm{E}[w_i]\|} \tag{24}$$

where κ^* is the point at which the variability response function takes its maximum value:

$$V_i(\kappa^*) \ge V_i(\kappa), -\infty < \kappa < \infty \tag{25}$$

5. Application of stochastic generalized differential quadrature to beam bending problem

Consider a prismatic beam of length L under a transversal load $q(x)$. The equation of equilibrium in the general case is:

$$\frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2 w}{\partial x^2}\right] = q(x) \tag{26}$$

where $E(x)$ is Young's modulus and $I(x)$ its moment of inertia. We assume that the flexural rigidity $E(x)I(x) = k(x)$ is a homogenous, one-dimensional random field with values at any grid point defined as:

$$k(x_i) = k_0[1 + \Delta_{k,i}] \tag{27}$$

where κ_0 is the expectation value, and $\Delta_{k,i}$ is the grid value of the zero-expectation, one-dimensional, homogenous random field $\Delta_k(x)$. The deterministic part of the stiffness matrix for GDQ method is then:

$$\mathbf{K}_0 = \left[k_i c_{ij}^{(4)} + 2\sum_{l=1}^{N} c_{il}^{(1)} k_l c_{ij}^{(3)} + \sum_{l=1}^{N} c_{il}^{(2)} k_l c_{ij}^{(2)} \right]_{i,j=1}^{N} \tag{28}$$

The stochastic part of the stiffness matrix is represented as a linear combination of N random variables $\Delta_{k,i}$

$$\Delta\mathbf{K}^{(e)} = \sum_{m=1}^{N} \Delta_{k,m} \Delta\mathbf{K}_m \tag{29}$$

where the coefficient matrices are given as follows:

$$\Delta\mathbf{K}_m = \left[\delta_{mi} c_{ij}^{(4)} + 2c_{im}^{(1)} c_{ij}^{(3)} + c_{im}^{(2)} c_{ij}^{(2)} \right]_{i,j=1}^{N} \tag{30}$$

The first-order approximation of the variance vector of the response deflection vector $\mathbf{w}$ is now:

$$\mathrm{Var}[\mathbf{w}] = \int_{-\infty}^{\infty} S_{kk}(k) \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{diag}(\mathbf{K}_0^{-1} \Delta\mathbf{K}_i \mathbf{w}_0) \mathbf{K}_0^{-1} \Delta\mathbf{K}_j \mathbf{w}_0 [CS]_{ij}(\kappa) d\kappa \tag{31}$$

where

$$[CS]_{ij}(\kappa) = \cos(\kappa x_i)\cos(\kappa x_j) + \sin(\kappa x_i)\sin(\kappa x_j) \tag{32}$$

If we consider a specific degree of freedom w_i, and its corresponding response variability component $V_i(\kappa)$, the related coefficient of variation is bounded as

$$\mathrm{COV}[w_i] \le \sigma_{kk} \frac{\sqrt{V_i(\kappa^*)}}{\|\mathrm{E}[w_i]\|} \tag{33}$$

where κ^* is the point at which the variability response function takes its maximum value.

6. Numerical examples

A cantilever beam with length L and flexural rigidity $E(x)\ I(x)$ under a concentrated load K on its free edge is taken as the first example (Fig. 1).

The variability response function is calculated for the deflection on the edge of the beam, $w(L)$. The results are evaluated using the stochastic differential quadrature method with nine grid points (Fig. 2). The coefficient of variation of the edge deflection is expressed as the function of the coefficient of variation of flexural rigidity.

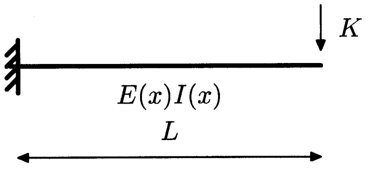

Fig. 1. Cantilever beam under a concentrated load on its free edge.

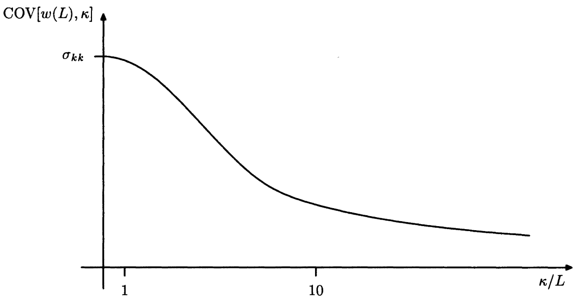

Fig. 2. Coefficient of variation of the response deflection w (L).

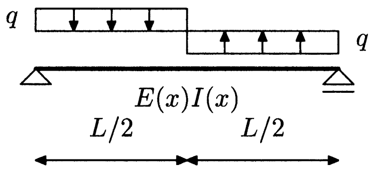

Fig. 3. Simply supported beam under an antisymmetric uniform load.

A simply supported beam with length L and flexural rigidity $E(x)I(x)$ subjected to an antisymmetric uniform load q is considered in the second example (Fig. 3). The variability response function is calculated for the deflection at the quarter-point, $w(L/4)$. The results are evaluated using the stochastic differential quadrature method with nine grid points (Fig. 4).

The maximum value of the variability response function is calculated to evaluate the spectral-distribution-free upper bound according to Eq. (24). The spectral-distribution-free upper bound is computed as follows:

$$\mathrm{COV}_{w(L/4)} \le 2.107\sigma_{kk} \tag{34}$$

The coefficient of variation takes its maximum value for $\kappa^* \approx 5.9L$. This example is very significant, and shows that the coefficient of variation of response deflection can become larger than the coefficient of variation of flexural rigidity of the beam.

7. Conclusions

The stochastic differential quadrature formulation is given for structural analysis problems with random material and/or geometric properties. The concept of variability response function is extended to the stochastic GDQ method. The stochastic stiffness matrix is expressed as a linear combination of deterministic and

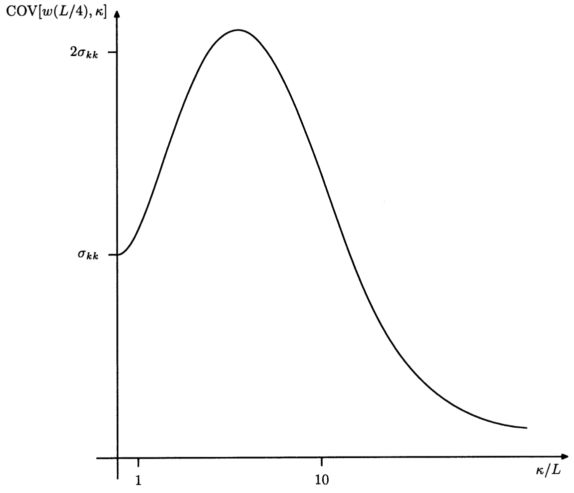

Fig. 4. Coefficient of variation of the response deflection ς (L/4).

stochastic parts. The first-order approximation of variability response function is used to describe the variability response. Spectral-distribution-free upper bounds have been computed using the concept of the variability response function. The introduced stochastic generalized differential quadrature formulation is described for a beam bending problem. The coefficient of variation of response deflection is expressed as a function of the coefficient of variation of flexural rigidity. It has been shown that the coefficient of variation of response deflection can become larger than the coefficient of variation of flexural rigidity.

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