FEM stress recovery using Trefftz polynomials

J. Mazúr*, V. Kompiš, P. Novák

University of Žilina, Faculty of Mechanical Engineering, Department of Mechanics, Univerzitná 1, 010 26 Žilina, Slovakia

Abstract

Stress recovery is an important part of the stress analysis in the FEM computations and can improve both accuracy and rate of convergence of computed stress fields. Satisfying both equilibrium and boundary conditions improves the results. In this contribution, a stress recovery technique using Trefftz (T-)interpolation polynomials and taking into account also the static boundary conditions (b.c.) in the most frequently used displacement FEM formulation is presented. A special kind of interpolation polynomial is used for elements on curved boundaries.

Keywords: Stress recovery; Trefftz polynomials; Curved boundaries; Displacement FEM; Accuracy; Convergence studies

1. Introduction

Stress recovery is an important part of the most frequently used displacement version of the FEM computation. It is known that stress smoothing using interpolation functions that satisfy the governing equations improves both the accuracy and the rate of convergence of the stress field computed in the second part (post-processing phase) of the FEM analysis [1,2]. The stresses are obtained by interpolation from the nodal displacements. If the T-functions (i.e. functions that satisfy the governing equations inside the domain, but not necessarily the boundary conditions) are used for the interpolation, the numerical procedure is especially efficient as the coefficients of T-polynomials are computed only once for each material [3,4]. The coefficients are obtained numerically [4]. Both these features increase the computational efficiency of the models. Inclusion of static b.c. is also very important and considerably improves the solution at the boundary points, where usually the worst accuracy is achieved in numerical solutions. General T-polynomials, however, are able to satisfy the b.c. only locally at a point. It is important to use such interpolation functions, which fit best the b.c. along the boundary. In this contribution, an application of the stress recovery technique is used to obtain the stress field in FEM models using T-polynomials. Additional interpolation functions are used to best fit the static b.c. for curved boundaries.

An implementation of the procedure was used to obtain stress fields from the nodal displacements in FEM programs Z88 [5] and ADINA [6].

* Corresponding author. Tel.: +421 41 5132965; Fax: +421 41 5652940; E-mail: mazuri@mppserv.utc.sk

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

2. Stress recovery from the nodal displacements and b.c.

We assume that the displacement field at a field point with the local co-ordinates **x**, with the local origin at the point where the stresses are to be computed (as the stresses are computed very simply from the coefficients of the first order polynomial terms), **u(x)**, is given in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathbf{c} \tag{1}$$

where **U(x)** is a matrix of T-displacement-functions and **c** is the vector of unknown coefficients. If T-polynomials are used for the T-functions, one can easy express strain and stress field from Eq. (1). The stress field can be written as

$$\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{S}(\mathrm{x})\mathbf{c} \tag{2}$$

where the matrix of T-stress-functions **S(x)** is obtained from the derivatives of the matrix **U(x)** and applying Hookes law in the usual way. Similarly, T-tractions are given as

$$\mathbf{t}(\mathrm{x}) = \mathbf{T}(\mathrm{x})\mathbf{c} \tag{3}$$

In this approximation the full T-polynomials of the chosen order are used as interpolators in all relations above. The T-displacements, **U(x)**, are in the form of polynomials, which satisfy the Lamé equations. The unknown coefficients **c** are to be computed for each point of interest (p.o.i.), where the stresses are evaluated from the patch of displacements and b.c., if the corresponding p.o.i. is on, or close to, the domain boundary.

The full polynomials of n-th order contain $2(2n + 1)$ in 2D and $3(n + 1)^2$ in 3D T-displacement functions. Each node and boundary point in the patch gives d equations for obtaining the unknown coefficients c, where d denotes the dimension. Generally, the patch is chosen so that the total number of equations is greater or equal to the number of unknown coefficients, c, and the problem is solved a LS sense as

$$\sum_{\mathrm{I}} w(A(\mathrm{x}_{\mathrm{I}})c - b_{\mathrm{I}})^2 = \min \tag{4}$$

where $\mathbf{u}_{\mathrm{I}}$ is the vector of displacements at the I-th nodal point and w is a weighting function that takes into account the dimensionality of corresponding equation.

Usually the largest errors in approximated fields are on the domain boundaries, or on the inter-domain boundaries between inhomogeneous material parts. The polynomial interpolation is efficient for stress computation at internal points and at points on straight boundaries. The shape of the boundaries, however, influences also the relation between the displacements and tractions at both the boundary points and the points close to them. The basic interpolation functions have to satisfy the b.c. not only at the p.o.i. but also in its close vicinity. The displacements and tractions on the convex boundaries can also be interpolated by polynomials, as the constant normal tractions result in a hydrostatic state of stress, and therefore the constant terms locally satisfy the b.c. in this region. However, in the domain with the concave boundaries, the term r^{-1} has to be added to both normal and tangent displacement components of the displacements, where r is the radius of curvature of the domain boundary at the corresponding p.o.i. These functions satisfy the Lamé equations (the polar coordinate form [7,8] is the simplest form for this particular case) and, thus, are T-functions. Moreover, they also give constant normal, or tangent, tractions along the local boundary, i.e. they are the lowest order terms that satisfy both the equilibrium equations and static b.c. at such p.o.i. on the domain boundary. The stresses are given with terms r^{-2} in this case. Especially, the first T-function is defined by the T-stresses $\sigma_{nn} = -\sigma_{tt} = 2G/r^2$ and $\sigma_{nt} = 0$ and corresponding T-displacements $u_n = 1/r$ and $u_t = 0$, and the second by the T-stresses $\sigma_{nn} = \sigma_{tt} = 0$ and $\sigma_{nt} = 1/r^2$ and corresponding T-displacements $u_n = 0$ and $u_t = 1/r$, where G is the material shear modulus. Similar T-functions have to be added in 3D, where double curvature also has to be considered.

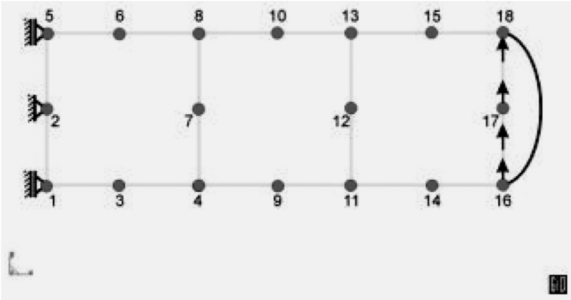

Fig. 1. 2D cantilever with shear load acting in the end section.

Notice that the discontinuity in the static b.c. can be correctly obtained in this way, which is very important by studying the local fields such as Hertz contact, etc.

3. Numerical experiments

In the first example, a 2D cantilever with shear load acting in the end section (Fig. 1) was modeled using three quadratic elements. The exact displacements are given by cubic polynomials, and so the displacements obtained by FEM analysis contain errors. The stresses obtained by the averaging technique are largest in their shear components at the nodes 7, 12 and 17 (the exact value there is 1.0000), and are equal to 0.8732, 0.8648 and 0.8754, respectively. The corresponding stresses obtained using cubic T-displacement for polynomials (i.e. 2×7 terms) are 0.9975, 1.0087 and 1.0000.

In the second example, a band with a hole (Fig. 2), with loads corresponding to the infinite plane with the hole in uniaxial tension in the x-direction, was analyzed using two different fine meshes. Von Mises stress fields obtained both by the averaging technique in FEM software and by T-interpolation from FEM nodal displacements are shown in Fig. 3.

4. Conclusions and future research in the field

The paper shows a simple and efficient technique to compute the smooth stress fields from the nodal displacement obtained from FEM analysis. Both displacements and stresses can be evaluated in arbitrary p.o.i. by interpolation from the displacements at discrete points of a patch of closest nodes using Trefftz interpolation functions and by including prescribed boundary conditions at the closest boundary point, if the corresponding p.o.i. is on, or close to, the boundary.

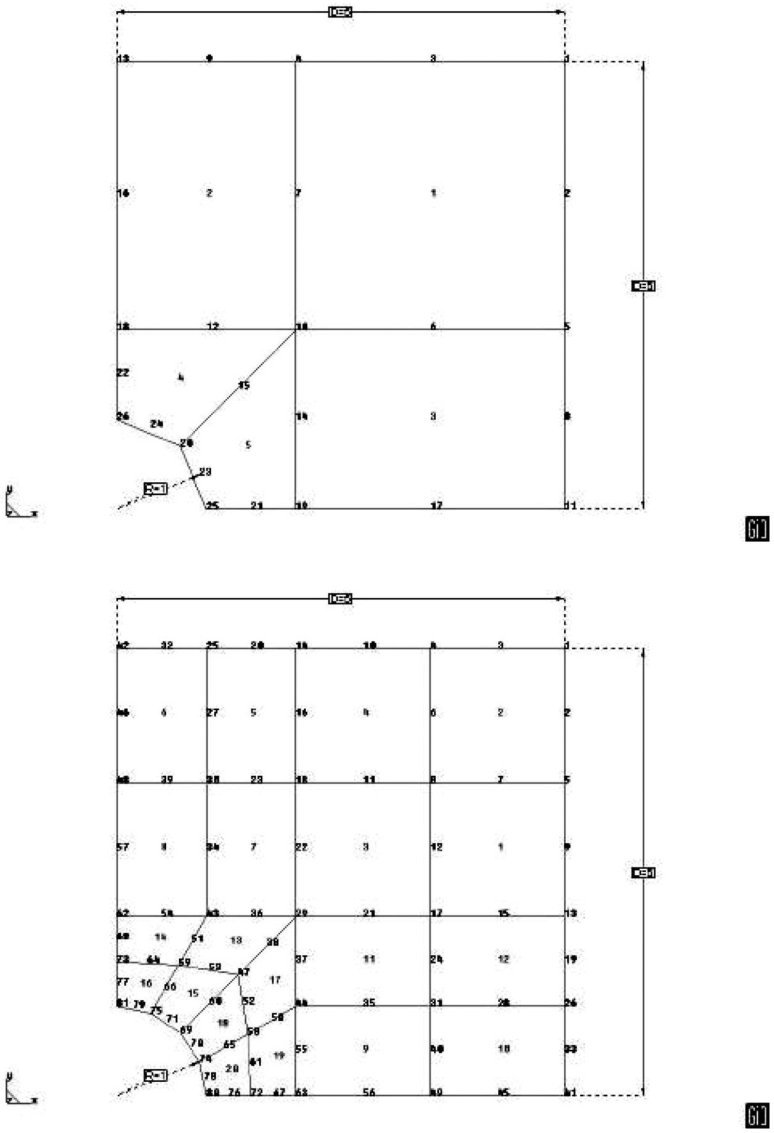

Fig. 2. A band with a hole.

Special T-functions are included for interpolation at the points on, or close to, concave boundaries.

More detailed study is planned into the application of the method to 3D problems with curved boundaries and also to the study of convergence and error estimation using this technique.

References

[1] Blacker T, Belytschko T. Superconvergent patch recovery with equilibrium and conjoint interpolant enhancements. Int J Numer Meth Eng 1994;37:517–536.

[2] Boroomand B, Zienkiewicz OC. Recovery by equilibrium in patches (REP). Int J Numer Meth Eng 1997;40:137–164.

[3] Kompiš V, Oravec J, Búry J. Reciprocity based FEM. In: Proc of the Conference on Numerical Methods in Continuum Mechanics, High Tatras, Slovakia, 1998.

[4] Kompiš V, Búry J. Stress smoothing in large strain, large rotation problems. In: Kompiš V, editor, Selected Topics

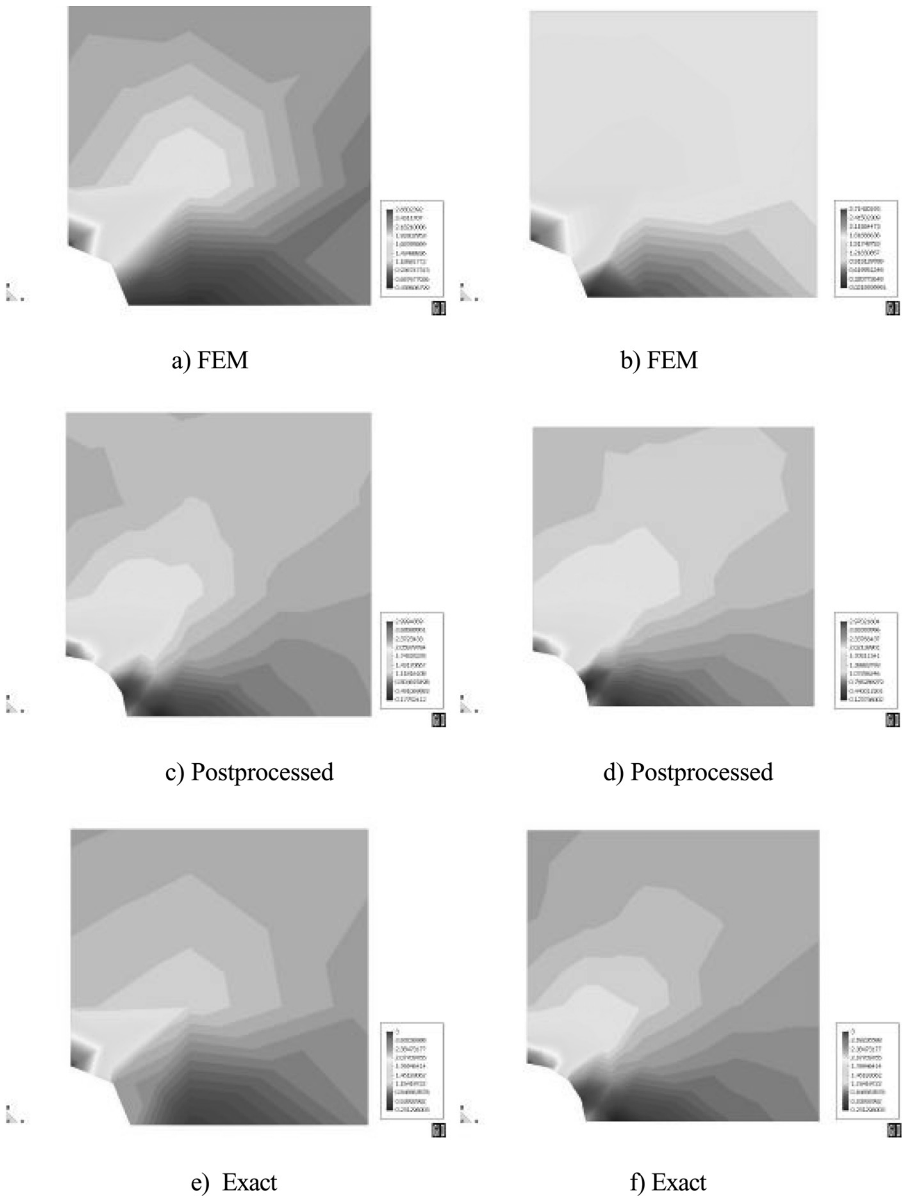

Fig. 3. Von Mises stress fields obtained by the averaging technique in FEM software and by T-interpolation from FEM nodal displacements.

in Boundary Integral Formulations for Solids and Fluids. Wien: Springer, 2002.
[5] Rieg F, Hackenschmidt R. Finite Element Analysis for Engineers. Munich: Hanser Verlag, 2003 (in German).
[6] Bathe KJ. Finite Element Procedures. Englewood Cliffs, NJ: Prentice Hall, 1996.
[7] Blokh VI. Theory of Elasticity. Kharkov: University Press, 1964 (in Russian).
[8] Kachanov M, Shafiro B, Tsukrov I. Handbook of Elasticity Solutions. Dordrecht: Kluwer Academic Publishers, 2003.