Positional description for nonlinear 2-D static and dynamic frame analysis by FEM with Reissner Kinematics

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Abstract

This paper presents a simple formulation to treat large deflections in 2-D frames by the finite element method. The present formulation does not use the concept of displacement; it considers position as the real variable of the problem. The strain determination is done directly from the proposed positional concept. A non-dimensional space is created and relative configurations are used to directly calculate the strain energy and its derivatives at general points. The initial configuration is assumed as the basis of calculation, i.e. Hooke's law relates reference stress and engineering strain. Reissner Kinematics are employed, i.e. initial plane cross-sections remain plane after deformation and angles are independent of the slope of central line. Some static and dynamic examples are presented in order to show the accuracy of the proposed technique.

Keywords: Geometric nonlinearity; Position description; Reissner Kinematics

1. Introduction

The increasing search for economy and optimal material application leads to the conception of very flexible structures. As a consequence, the equilibrium analysis in the non-deformed position is no more acceptable for most of applications.

Various researchers have presented important contributions regarding finite element procedures; see [1,2,3,4,5]. These researches are very important to the development of the human knowledge on conceptions on geometric non-linear analysis of structures.

In this study, as in Coda and Greco [6] for Bernoulli Kinematics, a simple position description language is used to present a geometrical non-linear formulation for Reissner Kinematics. This position description uses an intermediate non-dimensional space that allows defining a non-linear 'engineering' strain measure calculated from relative fiber length for different positions of the analyzed body.

The principle of minimum potential energy taken in Lagrangian Description is applied, considering a simple linear hyper-elastic constitutive relation. Reissner Kinematics are adopted, so the shear deformation effect in bending is considered. Examples show the good result of the proposed technique, other analytical and numerical solutions are used as reference.

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2. Positional formulation

For simplicity, only the static formulation is presented here.

The principle of minimum potential energy is written using position and considering conservative elastic problems as

$$\Pi = U_e - \mathrm{P} \tag{1}$$

where Π is the total potential energy, U_e is the strain energy and P is the potential energy of the applied concentrated forces.

The strain energy can be written for the reference volume V_0 as

$$U_e = \int_{V_0} u_e dV_0 = \int_{V_0} \frac{1}{2}\sigma_e \varepsilon_e dV_0 \tag{2}$$

where σ_e is defined here as the 'engineering stress', i.e.

the energy conjugate of the proposed 'non-linear engineering strain' ε_e by Coda and Greco [6]. The potential energy of applied forces is written as

$$P = F_j \Psi_j \tag{3}$$

where Ψ is the set of positions independent of each other, which may be occupied by a point of the body. The total potential energy is written as

$$\Pi = \int_{V_0} \frac{1}{2}\sigma_e \varepsilon_e dV_0 \; - F_j \Psi_j \tag{4}$$

In order to perform numerically the integral indicated in Eq. (4), it is necessary to map the geometric approximation of the body in study and to know its relation with the strain measurement adopted. Figure 1 shows the reference configuration (B_0), the actual configuration (B_1) and the non-dimensional space that provides a 'bridge' between B_0 and B_1. The reference configuration adopted is the non-deformed position of body. In Fig. 1, A^I is the gradient of the mapping between the non-dimensional space and B_0 and A^{II} is the same for B_1.

A generic point position *(x, y)* inside the element, for B_0 and B_1, can be written as function of the non-dimensional coordinates, i.e.

$$(x^i, y^i) = (x^i_m, y^i_m) + \frac{h}{2}\eta(-\sin\theta, \cos\theta)^i \tag{5}$$

where (x^i_m, y^i_m) is a point in the central line, h is the element width, $(-\sin\theta, \cos\theta)^i$ is a unity vector in the plane of a generic cross section, and η is one of the non-dimensional coordinates.

By deriving Eq. (5) regarding the non-dimensional space coordinates, the elements of A_{I} and A_{II} can be written as

$$A_i = \begin{bmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{bmatrix} = \begin{bmatrix} \frac{dx^i}{d\xi} & \frac{dx^i}{d\eta} \\ \frac{dy^i}{d\xi} & \frac{dy^i}{d\eta} \end{bmatrix} \tag{6}$$

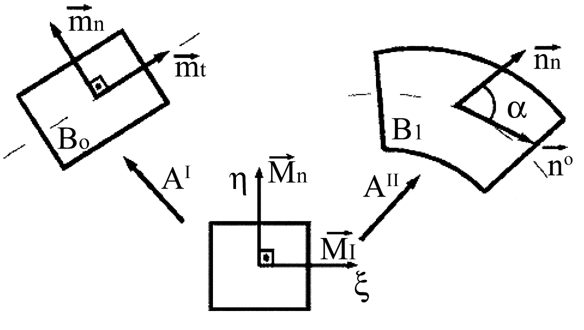

Fig. 1. Auxiliary non-dimensional space and simple mapping.

where,

$$A^i_{11} = \frac{dx_m}{d\xi} - \frac{h}{2}\eta\frac{d\theta}{d\xi}\cos\theta \tag{7}$$

$$A^i_{12} = -\frac{h}{2}\sin\theta \tag{8}$$

$$A^i_{21} = \frac{dx_m}{d\xi} - \frac{h}{2}\eta\frac{d\theta}{d\xi}\sin\theta \tag{9}$$

$$A^i_{12} = \frac{h}{2}\cos\theta \tag{10}$$

The index 'i', Eq. (5) through Eq. (10), denotes the number of the transformation, i.e. $i = \mathrm{I} \Rightarrow$ non-dimensional space to reference configuration (B_0) and $i = \mathrm{II} \Rightarrow$ non-dimensional space to current configuration (B_1).

The mapping stretches λ^i_t and λ^i_n can be calculated by following the reference directions ξ and η, defined here by the unit vectors $\vec{M}_t = [1,0]^T$ and $\vec{M}_n = [0,1]^T$ shown in Fig. 1 as follows:

$$\lambda^i_t = \lambda(\vec{M}_t) = \left| A^i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right| = \sqrt{(A^i_{11})^2 + (A^i_{21})^2} \tag{11}$$

$$\lambda^i_n = \lambda(\vec{M}_n) = \left| A^i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right| = 1 \tag{12}$$

In this formulation the third direction is considered non-deformable and therefore the principal direction $\lambda_3 = 1$. The angle α (Fig. 1) between vectors $\vec{m}_t$ and $\vec{m}_n$, after deformation, is easily calculated, as

$$\alpha = \arccos\left\{ \frac{[1\ 0][A^T_{II} A_{II}]\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\lambda^{II}_t \lambda^{II}_n} \right\} = \arccos\left\{ \frac{A^{II}_{11} A^{II}_{12} + A^{II}_{21} A^{II}_{22}}{\sqrt{(A^{II}_{11})^2 + (A^{II}_{21})^2}} \right\} \tag{13}$$

The distortion occurring when the body changes from configuration B_0 to B_1 can be simply calculated as follows:

$$\gamma_{nt} = \alpha - \frac{\pi}{2} \tag{14}$$

And the stretches λ_t and λ_n from B_0 to B_1 are given by

$$\lambda_t = \frac{\lambda^{II}_t}{\lambda^I_t} = \frac{\sqrt{(A^{II}_{11})^2 + (A^{II}_{21})^2}}{\sqrt{(A^I_{11})^2 + (A^I_{21})^2}} \tag{15}$$

$$\lambda_n = \frac{\lambda^{II}_n}{\lambda^I_n} = 1 \tag{16}$$

Finally, the engineering strains are calculated as

$$\varepsilon_t = \lambda_t - 1 \tag{17}$$

$$\varepsilon_n = \lambda_n - 1 = 0 \tag{18}$$

where ε_t and ε_n are the 'non-linear engineering strains' related, respectively, to $\vec{T}$ and $\vec{N}$ directions in the initial configuration, B_0. Therefore, the specific strain energy for a simple linear constitutive law, is given by

$$u_e = \frac{1}{2}E\left(\varepsilon_t^2 + \frac{\gamma_{tn}^2}{2}\right) \tag{19}$$

where E is the young modulus of the material.

Using Eq. (19) into Eq. (2), the integral of the total strain energy becomes

$$U_e = \int_{V_0} \frac{1}{2}E\left(\varepsilon_t^2 + \frac{\gamma_{tn}^2}{2}\right) dV_0 \tag{20}$$

Therefore, the total potential energy of the system (see Eq. (4)) can be written as follows:

$$\Pi = \int_{V_0} \frac{1}{2}E\left(\varepsilon_t^2 + \frac{\gamma_{tn}^2}{2}\right) dV_0 \; - F_j\Psi_j \tag{21}$$

where V_0 is the volume of the reference configuration B_0.

The strategy for minimizing Eq. (21) employing FEM is performed in the next section.

It is important to emphasize that $\frac{\partial u_e}{\partial \varepsilon_t}$ gives σ_t and $\frac{\partial u_e}{\partial \gamma_{tn}}$ gives, τ_{tn}, which are the nominal or engineering stress components.

3. The numerical method

The proposed technique is similar to any finite element method except the determination of strain, where the auxiliary space is used in an original fashion.

Dividing the body into finite elements the variables in Eqs. (9) (10) along the central line of the element, i.e. x_m, y_m and θ are approximated as follows:

$$x_m = \Phi_i X_i \tag{27}$$

$$y_m = \Phi_i Y_i \tag{28}$$

$$\theta = \Phi_i \Theta_i \tag{29}$$

where the vectors X, Y and Θ are nodal variables and vector Φ are the set of shape functions. Besides, the angle θ is not the first derivative of the transversal displacement as in the Euler–Bernoulli hypothesis.

Substituting Eqs. (27), (28) and (29) in Eqs. (9) (10), it becomes:

$$x = \Phi_i X_i - \frac{h}{2}\eta \sin(\Phi_i \Theta_i) \tag{30}$$

$$y = \Phi_i Y_i + \frac{h}{2}\eta \cos(\Phi_i \Theta_i) \tag{31}$$

In the same way, substituting the variables approximations (Eq. (27) to Eq. (29)) in gradient mapping expressions, i.e. Eq. (12) to Eq. (15), gives:

$$A_{11}^i = \beta_j X_j^i - \frac{h}{2}\eta\left(\beta_j \Theta_j^i\right)\cos\left(\Phi_k \Theta_k^i\right) \tag{32}$$

$$A_{12}^i = -\frac{h}{2}\eta \sin\left(\Phi_k \Theta_k^i\right) \tag{33}$$

$$A_{21}^i = \beta_j Y_j^i - \frac{h}{2}\eta\left(\beta_j \Theta_j^i\right)\sin\left(\Phi_k \Theta_k^i\right) \tag{34}$$

$$A_{12}^i = \frac{h}{2}\eta \cos\left(\Phi_k \Theta_k^i\right) \tag{35}$$

where,

$$\beta_j = \frac{d\Phi_j}{d\xi} \tag{36}$$

The Total Potential Energy in Eq. (26) can be expressed as a function of the position coordinates, that is, $\Pi \rightarrow f(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2, X_3, Y_3, \Theta_3)$. Therefore, the strategy is to minimize Eq. (26) related to the nodal parameters in order to find out the equilibrium position of the body.

Or, in compact notation,

$$\frac{\partial \Pi}{\partial p_i} = g_i(p_j) = f_i(p_j) - F_i = 0 \tag{37}$$

where p_i is the vector of the nodal position.

In vector representation one has:

$$g(p, F) = 0 \tag{38}$$

The vector function $g(p)$ is non-linear regarding the nodal parameters (p and F). To solve Eq. (38) one can use the Newton-Raphson procedure.

In the next section, numerical examples are shown and compared to the literature in order to validate the proposed technique.

4. Numerical examples

4.1. Pinned fixed diamond frame

The following properties are adopted to run the problem: L = 1, E = 1, I = 1 and A = 1000. Figure 2 shows the diamond frames and the measured 'displacements'. Displacements are calculated here by the difference between positions to allow comparisons with Mattiasson [7]. Symmetry is considered.

Twenty finite elements were used to run this problem. The maximum achieved errors are 1% in tension and 0.3% in compression. There is no measurable difference between the results obtained here and those obtained by Coda and Greco [6].

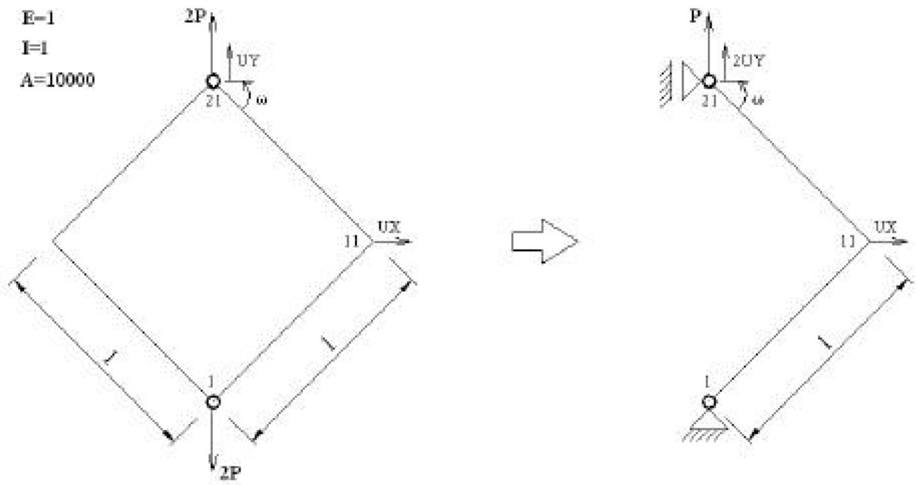

Fig. 2. Diamond frame, static scheme for tension ($P > 0$) and compression ($P < 0$) situations.

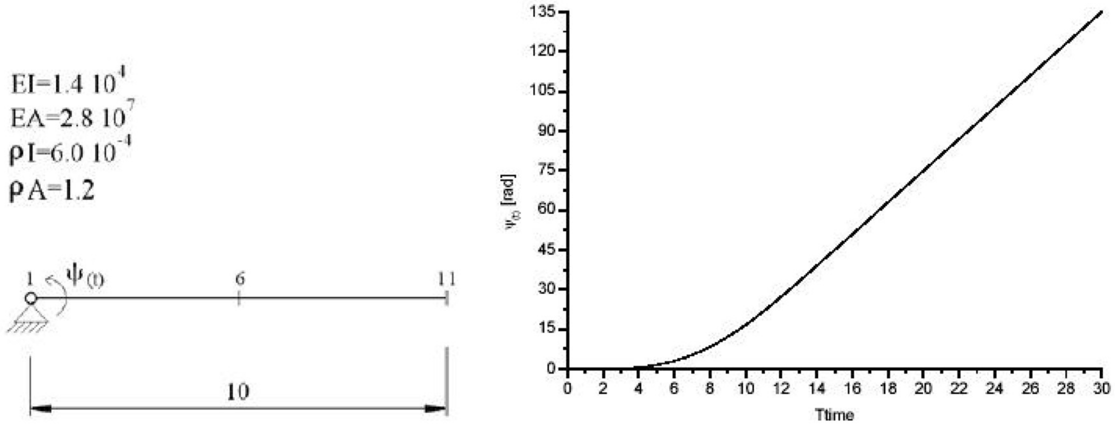

Fig. 3. Flexible spin-up maneuver input data.

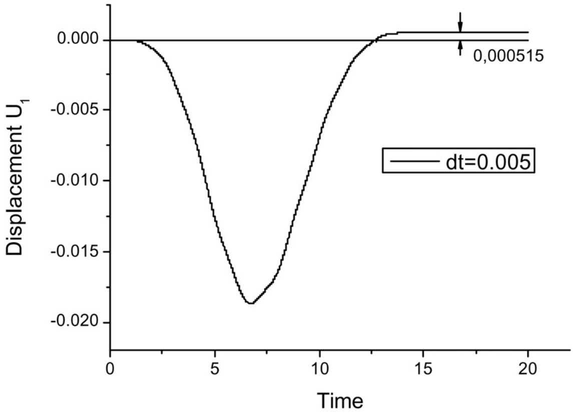

Fig. 4. Displacement U_1.

4.2. Spin-up maneuver

The second numerical example is a simple fixed flexible beam and is a benchmark of non-linear dynamic formulations (see Fig. 3). Ten finite elements are used in the discretization. In Fig. 4 the displacement U_1 is shown. A consistent mass matrix is used to solve this problem.

5. Conclusions

In this paper a consistent and simple formulation is proposed to solve geometrically non-linear plane frame problems considering Reissner Kinematics. In order to show the didactic possibilities of the technique, a simple engineering language is used. The analyzed examples, mainly the second, show that shear effects are of less importance for the analysis of very deformable structures, but independent rotating and Cartesian position approximations, achieved using Reissner Kinematics, bring an important improvement to the non-linear analysis of frames by the position description method. The formulation developed can be used for practical applications and presents good convergence and accuracy.

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