Mode III analysis of an interface crack in a bimaterial wedge made from magneto-electro-elastic media

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Abstract

The mode III analysis of an interface crack in a bimaterial magneto-electro-elastic wedge has been studied. The three-phases magneto-electro-elastic field is induced by the piezoelectric/piezomagnetic $BaTiO_3$-$CoFe_2O_4$ composite materials. For the crack problems, the intensity factors at crack tips are derived analytically. Also, the energy density criterion is applied to predict the fracture behavior of the interface crack.

Keywords: Magneto-electro-elastic; Wedge; Interface crack

1. Introduction

The piezoelectric/piezomagnetic $BaTiO_3$-$CoFe_2O_4$ composite material performs the magneto-electro-elastic (MEE) properties [1,2]. In recent years, the mode III (antiplane) fracture analyses of the MEE materials have been studied [3,4]. The elastic analysis of a bimaterial wedge with an interface crack has been studied [5,6]. Furthermore, Chue and Liu [7] solved the same wedge structure made from piezoelectric materials. In this paper, we try to solve the MEE field of a bimaterial $BaTiO_3$-$CoFe_2O_4$ wedge containing an interface crack. The intensity factors of stress, strain, electric displacement, electric field, magnetic induction and magnetic field at crack tips are derived analytically. The energy density criterion is applied to predict the fracture behavior of the interface crack.

2. Basic formulations

Figure 1 shows a bimaterial wedge composed of two bonded $BaTiO_3$-$CoFe_2O_4$ materials with same wedge angle α. An interface crack AB locates on the common edge between $r = a$ and $r = b$. The antiplane shearing forces F, inplane surface charges Q and inplane magnetic inductions B_c are applied on the edges at a distance $r = h$. Since the piezoelectric/piezomagnetic materials are transversely isotropic and polarized in the z-direction, the antiplane elastic field coupled with the inplane electric and magnetic fields is considered. The field quantities include the shear stresses (σ_{rz}, $\sigma_{\theta z}$), shear strains (γ_{rz}, $\gamma_{\theta z}$), displacement (w), electric displacements (D_r, D_θ), electric fields (E_r, E_θ), electric potential (ϕ), magnetic inductions (B_r, B_θ), magnetic fields (H_r, H_θ) and magnetic potential (ψ). The constitutive equations of the problem are as follows:

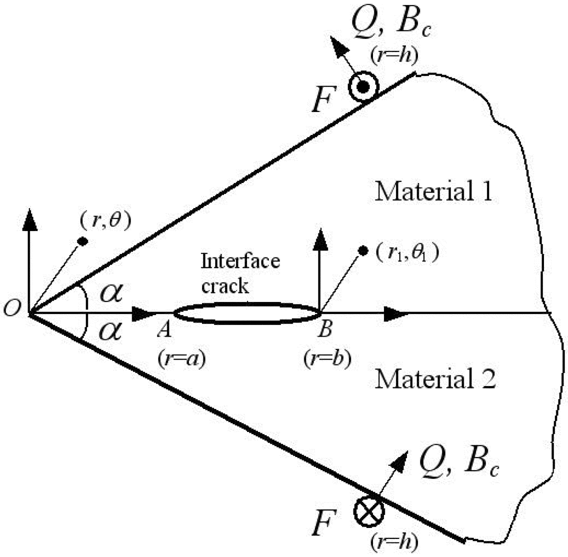

Fig. 1. A bimaterial MEE wedge with an interface crack.

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$$\begin{bmatrix} \sigma_{\theta z}^{(i)} \\ \sigma_{rz}^{(i)} \\ D_r^{(i)} \\ D_\theta^{(i)} \\ B_r^{(i)} \\ B_\theta^{(i)} \end{bmatrix} = \begin{bmatrix} c_{44}^{(i)} & 0 & 0 & -e_{15}^{(i)} & 0 & -q_{15}^{(i)} \\ 0 & c_{44}^{(i)} & -e_{15}^{(i)} & 0 & -q_{15}^{(i)} & 0 \\ 0 & e_{15}^{(i)} & \varepsilon_{11}^{(i)} & 0 & \lambda_{11}^{(i)} & 0 \\ e_{15}^{(i)} & 0 & 0 & \varepsilon_{11}^{(i)} & 0 & \lambda_{11}^{(i)} \\ 0 & q_{15}^{(i)} & \lambda_{11}^{(i)} & 0 & \Gamma_{11}^{(i)} & 0 \\ q_{15}^{(i)} & 0 & 0 & \lambda_{11}^{(i)} & 0 & \Gamma_{11}^{(i)} \end{bmatrix} \begin{bmatrix} \gamma_{\theta z}^{(i)} \\ \gamma_{rz}^{(i)} \\ E_r^{(i)} \\ E_\theta^{(i)} \\ H_r^{(i)} \\ H_\theta^{(i)} \end{bmatrix}, \quad i = 1,2 \tag{1}$$

where c_{44}, ε_{11}, e_{15}, Γ_{11}, q_{15} and λ_{11} are the elastic stiffness constant, dielectric constant, piezoelectric coefficient, magnetic permeability, piezomagnetic coefficient and magnetoelectric coefficient, respectively. The superscript i denotes materials 1 and 2. For static analysis, λ_{11} should be zero. The governing equations and the solutions for w, ϕ and ψ are:

$$c_{44}^{(i)}\nabla^2 w^{(i)} + e_{15}^{(i)}\nabla^2\phi^{(i)} + q_{15}^{(i)}\nabla^2\psi^{(i)} = 0$$
$$e_{15}^{(i)}\nabla^2 w^{(i)} - \varepsilon_{11}^{(i)}\nabla^2\phi^{(i)} = 0,\ q_{15}^{(i)}\nabla^2 w^{(i)} - \Gamma_{11}^{(i)}\nabla^2\psi^{(i)} = 0 \tag{2}$$

$$\nabla^2 w^{(i)} = 0, \quad \nabla^2\phi^{(i)} = 0, \quad \nabla^2\psi^{(i)} = 0 \tag{3}$$

The boundary conditions on edges $\theta = \pm\alpha$ are as follows:

$$\sigma_{\theta z}^{(1)}(r,\alpha) = F\delta(r-h), \quad \sigma_{\theta z}^{(2)}(r,-\alpha) = F\delta(r-h) \tag{4}$$

$$D_\theta^{(1)}(r,\alpha) = Q\delta(r-h), \quad D_\theta^{(2)}(r,-\alpha) = Q\delta(r-h) \tag{5}$$

$$B_\theta^{(1)}(r,\alpha) = B_c\delta(r-h), \quad B_\theta^{(2)}(r,-\alpha) = B_c\delta(r-h) \tag{6}$$

where δ is the Dirac-Delta function. For the generality of mathematical derivation, the distance h satisfies $a \le b \le h$. The continuity conditions along the interface are:

$$w^{(1)}(r,0) = w^{(2)}(r,0), \phi^{(1)}(r,0) = \phi^{(2)}(r,0), \psi^{(1)}(r,0) = \psi^{(2)}(r,0),\ \sigma_{\theta z}^{(1)}(r,0) = \sigma_{\theta z}^{(2)}(r,0)$$
$$D_\theta^{(1)}(r,0) = D_\theta^{(2)}(r,0)$$
$$B_\theta^{(1)}(r,0) = B_\theta^{(2)}(r,0) \quad 0 \le r \le a \quad b \le r < \infty \tag{7}$$

The conditions on the crack surfaces, which are impermeable and free of traction, can be written as:

$$\sigma_{\theta z}^{(1)}(r,0) = \sigma_{\theta z}^{(2)}(r,0) = 0, D_\theta^{(1)}(r,0) = D_\theta^{(2)}(r,0) = 0,$$
$$B_\theta^{(1)}(r,0) = B_\theta^{(2)}(r,0) = 0 \quad a \le r \le b \tag{8}$$

Three functions $f(r)$, $g(r)$ and $l(r)$ are defined for the crack surfaces [5]:

$$f(r) = \frac{\partial}{\partial r}\left[w^{(1)}(r,+0) - w^{(2)}(r,-0)\right], \quad f(r) = 0 \text{ in } 0 \le r \le a \text{ and } b \le r < \infty \tag{9}$$

$$g(r) = \frac{\partial}{\partial r}\left[\phi^{(1)}(r,+0) - \phi^{(2)}(r,-0)\right], \quad g(r) = 0 \text{ in } 0 \le r \le a \text{ and } b \le r < \infty \tag{10}$$

$$l(r) = \frac{\partial}{\partial r}\left[\psi^{(1)}(r,+0) - \psi^{(2)}(r,-0)\right], \quad l(r) = 0 \text{ in } 0 \le r \le a \text{ and } b \le r < \infty \tag{11}$$

These functions are non-zero in $a \le r \le b$. Also, the single-valuedness conditions of the displacement, electric potential and magnetic potential require that

$$\int_a^b f(r)dr = 0, \text{ provided that } \int_a^b g(r)dr = 0, \int_a^b l(r)dr = 0 \tag{12}$$

3. Solutions

Applying the Mellin transform on the first equation of Eq. (3), gives

$$\frac{d^2\bar{w}^{(i)}}{d\theta^2} + p^2\bar{w}^{(i)} = 0, \text{ provided that} \left[r^{p+1}\frac{\partial w^{(i)}}{\partial r} - pr^p w^{(i)}\right]_0^\infty = 0 \tag{13}$$

where $\bar{w}$ denotes the transformed quantity and p is a complex. The solutions of $\bar{w}$ in Eq. (13) can be obtained. Similarly, we can also obtain the solutions of ϕ and ψ. The solutions are:

$$\bar{w}^{(1)} = C_1(p)\cos p\theta + C_2(p)\sin p\theta,$$
$$\bar{w}^{(2)} = C_7(p)\cos p\theta + C_8(p)\sin p\theta \tag{14}$$

$$\bar{\phi}^{(1)} = C_3(p)\cos p\theta + C_4(p)\sin p\theta,$$
$$\overline{\phi}^{(2)} = C_9(p)\cos p\theta + C_{10}(p)\sin p\theta \tag{15}$$

$$\bar{\psi}^{(1)} = C_5(p)\cos p\theta + C_6(p)\sin p\theta,$$
$$\overline{\psi}^{(2)} = C_{11}(p)\cos p\theta + C_{12}(p)\sin p\theta \tag{16}$$

where $C_i(p)$ (i = 1, 2, 3, ... , 12) can be deduced from Eqs. (4)–(11). Applying the inverse Mellin transform on Eqs. (14)–(16) and using the residue theorem and appropriate path of integration [6], w, ϕ and ψ are obtained. Only the region $a \le r \le b$ in material 1 is considered to compute the intensity factors at the crack tips. After using Eqs. (8)–(12) and Eqs. (14)–(16) with tedious mathematical procedures [6,7] and the singular integral equation, $f(r)$, $g(r)$ and $l(r)$ are obtained. Furthermore, the stress, electric displacement and magnetic induction are obtained as follows:

$$\sigma_{\theta z}^{(1)}(r,\theta) = \frac{-F}{r}[X_1 + X_2], \quad D_\theta^{(1)}(r,\theta) = \frac{-Q}{r}[X_1 + X_2],$$
$$B_\theta^{(1)}(r,\theta) = \frac{-B_c}{r}[X_1 + X_2],\ a \le r \le b \tag{17}$$

where

$$X_1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\frac{r}{h}\right)^{p_n} p_n \cos(p_n\theta),$$

$$X_2 = \sum_{n=1}^{\infty}\frac{\left(\int_a^r \left(\frac{\nu}{r}\right)^{p_n} f*(\nu)d\nu - \int_r^b \left(\frac{\nu}{r}\right)^{-p_n} f^*(\nu)d\nu\right)\sin(p_n\alpha - p_n\theta)}{\alpha\sin(p_n\alpha)} \tag{18}$$

$$p_{\pm n} = \pm\frac{2n-1}{2\alpha}\pi\,, \quad n = 1, 2, \cdots, \infty \tag{19}$$

$$f^*(r) = \frac{1}{\alpha}\sqrt{h^\varsigma(a^\varsigma + h^\varsigma)(b^\varsigma + h^\varsigma)}\left[\frac{1}{r^\varsigma + h^\varsigma} - \frac{1}{b^\varsigma + h^\varsigma}\right.$$
$$\left.\times\frac{\prod(k, m, \pi/2)}{K(k, \pi/2)}\right]\left(r^{(\zeta/2)-1}\right)\frac{1}{\sqrt{(r^\varsigma - a^\varsigma)(b^\varsigma - r^\varsigma)}}, \; a \le r \le b \tag{20}$$

$$\prod(k, m, \pi/2) = \int_0^{\pi/2}\frac{d\theta}{(1 + m\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}$$

$$K(k, \pi/2) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{21}$$

$$k^2 = \frac{d-c}{d}, \; m = -\frac{d-c}{d+e}, \quad \zeta = \pi/\alpha, \; c = a^\varsigma, \; d = b^\varsigma, \; e = h^\varsigma \tag{22}$$

The function $f^*(r)$ is independent of material properties. Equations (17) indicate that the stress, electric displacement and magnetic induction are uncoupled and are independent of material properties. This conclusion is true only if the geometry, external loadings and the impermeable crack condition are symmetric with respect to the interface. The same conclusion can be seen in several previous studies [4,7,8].

Since the uncoupled phenomenon and independency of material properties in Eqs. (17), we define the stress, electric displacement and magnetic induction intensity factors at both crack tips ($r = a$ and $r = b$) from the concepts in [5–7]:

$$K^\sigma_{III}(a) = \lim_{r\to a+}\sqrt{2\pi(r-a)}Ff^*(r),$$
$$K^\sigma_{III}(b) = -\lim_{r\to b-}\sqrt{2\pi(b-r)}Ff^*(r) \tag{23}$$

$$K^D_{III}(a) = \lim_{r\to a+}\sqrt{2\pi(r-a)}Qf^*(r),$$
$$K^D_{III}(b) = -\lim_{r\to b-}\sqrt{2\pi(b-r)}Qf^*(r) \tag{24}$$

$$K^B_{III}(a) = \lim_{r\to a+}\sqrt{2\pi(r-a)}B_cf^*(r),$$
$$K^B_{III}(b) = -\lim_{r\to b-}\sqrt{2\pi(b-r)}B_cf^*(r) \tag{25}$$

From Eq. (1), the strain, electric field and magnetic field intensity factors become:

$$K^{\gamma(i)}_{III}(a) = \frac{Qe^{(i)}_{15}\Gamma^{(i)}_{11} + B_cq^{(i)}_{15}\varepsilon^{(i)}_{11} + F\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}{e^{(i)^2}_{15}\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to a+}\sqrt{2\pi(r-a)}f^*(r),$$

$$K^{\gamma(i)}_{III}(b) = -\frac{Qe^{(i)}_{15}\Gamma^{(i)}_{11} + B_cq^{(i)}_{15}\varepsilon^{(i)}_{11} + F\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}{e^{(i)^2}_{15}\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to b-}\sqrt{2\pi(b-r)}f^*(r)$$

$$K^{E(i)}_{III}(a) = \frac{-B_ce^{(i)}_{15}q^{(i)}_{15} + Q(q^{(i)}_{15}2 + c^{(i)}_{44}\Gamma^{(i)}_{11}) - Fe^{(i)}_{15}\Gamma^{(i)}_{11}}{e^{(i)^2}_{15}\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to a+}\sqrt{2\pi(r-a)}f^*(r),$$

$$K^{E(i)}_{III}(b) = -\frac{-B_ce^{(i)}_{15}q^{(i)}_{15} + Q(q^{(i)}_{15}2 + c^{(i)}_{44}\Gamma^{(i)}_{11}) - Fe^{(i)}_{15}\Gamma^{(i)}_{11}}{e^{(i)^2}_{15}\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to b-}\sqrt{2\pi(b-r)}f^*(r)$$

$$K^{H(i)}_{III}(a) = \frac{B_c(e^{(i)^2}_{15} + c^{(i)}_{44}\varepsilon^{(i)}_{11}) - q^{(i)}_{15}(Qe^{(i)}_{15} + F\varepsilon^{(i)}_{11})}{e^{(i)}_{15}2\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to a+}\sqrt{2\pi(r-a)}f^*(r),$$

$$K^{H(i)}_{III}(b) = -\frac{B_c(e^{(i)^2}_{15} + c^{(i)}_{44}\varepsilon^{(i)}_{11}) - q^{(i)}_{15}(Qe^{(i)}_{15} + F\varepsilon^{(i)}_{11})}{e^{(i)^2}_{15}\Gamma^{(i)}_{11} + q^{(i)^2}_{15}\varepsilon^{(i)}_{11} + c^{(i)}_{44}\Gamma^{(i)}_{11}\varepsilon^{(i)}_{11}}\lim_{r\to b-}\sqrt{2\pi(b-r)}f^*(r) \tag{26}$$

where i denotes the material i. Equations (26) show the coupling effects and can be reduced to the piezoelectric problem in [7] when the magnetic and piezomagnetic effects are ignored.

The energy density theory [3] is employed for studying the crack problem in piezoelectric or piezoelectric/piezomagnetic materials. In this study, the energy density factor is defined as:

$$S^{(i)} = \frac{1}{4\pi}\left[K^\sigma_{III}K^{\gamma(i)}_{III} + K^D_{III}K^{E(i)}_{III} + K^B_{III}K^{H(i)}_{III}\right] \tag{27}$$

Note that $S^{(i)}$ for materials 1 or 2 are independent of θ_1 (local coordinate system at the crack tip) for this specific antiplane problem. This conclusion is similar to that of the elastic material. The crack will propagate along the direction of the least fracture resistance S_c.

4. Numerical results and discussions

The bimaterial $BaTiO_3$-$CoFe_2O_4$ wedge with $a = 0.01$ m and $b - a = 0.01$ m is considered. The mixture rule [3] for the $BaTiO_3$-$CoFe_2O_4$ composite is:

$$\kappa^C_{ij} = \kappa^I_{ij}V_f + \kappa^M_{ij}(1 - V_f) \tag{28}$$

where κ_{ij} is the material constant and V_f is the volume fraction of $BaTiO_3$. The superscripts C, I and M represent the composite, inclusion and matrix, respectively. In this case, the volume fractions of $BaTiO_3$ for materials 1 and 2 are 50% and 30%, respectively. Table 1 shows the material constants of $BaTiO_3$ and $CoFe_2O_4$ [9]. Although it was used in many studies, we think the negative value, $\Gamma_{11} = -590 \times 10^{-6}(Ns^2/C^2)$, of $CoFe_2O_4$ is questionable. Pan [10] also didn't use this value due to the negative internal energy for the Stroh formalism. The handbook [11] indicates that the magnetic permeability of ferrimagnetic materials such as $CoFe_2O_4$ should be positive. Due to the lack of correct data in

Table 1
Material constants of $BaTiO_3$ and $CoFe_2O_4$ [9]

Material constants	$BaTiO_3$	$CoFe_2O_4$
c_{44} (N/m^2)	43×10^9	45.3×10^9
e_{15} (C/m^2)	11.6	0
q_{15} (N/ Am)	0	550
ε_{11} (C/ Vm)	11.2×10^{-9}	0.08×10^{-9}
Γ_{11} (Ns^2/ C^2)	5×10^{-6}	-590×10^{-6}*

* We assume $\Gamma_{11} = 100 \times 10^{-6}$ Ns^2/ C^2 in our numerical calculation.

past references, we assign a value of Γ_{11} for $CoFe_2O_4$, say 100×10^{-6} Ns^2/C^2.

To investigate the effects of the wedge angle α on $S^{(i)}$, the wedge is subjected to $F = 10$ N/m, $Q = 1 \times 10^{-8}$ C/m and $B_c = 1 \times 10^{-5}$ Wb/m at edges with a distance $h = 0.03$ m. The variations of $S^{(i)}(a)$ and $S^{(i)}(b)$ at both crack tips with α are shown in Fig. 2. It shows $S^{(1)}(a) > S^{(2)}(a)$ and $S^{(1)}(b) > S^{(2)}(b)$. We find that $S^{(i)}(a) > S^{(i)}(b)$ for $\alpha > 0.37\ \pi$ and $S^{(i)}(a)$ reaches its maximum when $\alpha = 0.37\ \pi$. However, when $\alpha < 0.37\ \pi$, $S^{(i)}(b)$ reaches its maximum and is greater than $S^{(i)}(a)$.

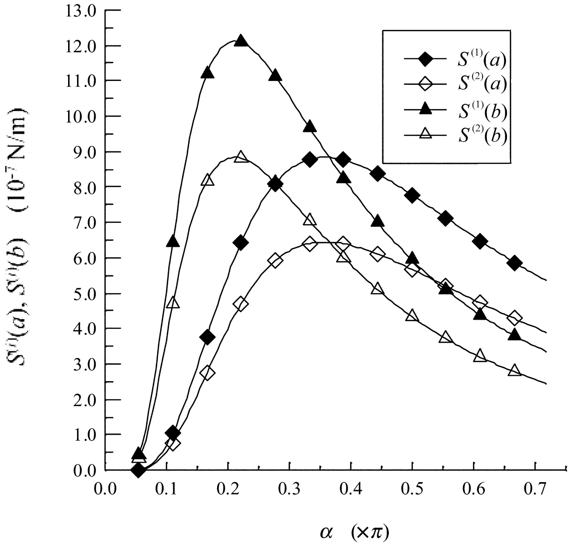

Fig. 2. The variations of energy density factors with the wedge angle α.

5. Conclusions

The intensity factors and energy density factors are obtained to predict the fracture behavior of the interface crack in a bimaterial MEE wedge. The energy density factor for the antiplane problem is independent of θ_1. The crack will propagate along the direction of the least fracture resistance S_c. From the numerical results, the effects of the wedge angle have been discussed.

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