Adaptive variational multiscale methods for elliptic problems

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Abstract

The variational multiscale method provides a framework for construction of adaptive multiscale finite element methods. A new adaptive finite element method is presented based on the variational multiscale method and an a posteriori error estimate in the energy norm for this method. The estimate captures crucial parameters of the method and shows how they are related. An adaptive algorithm is presented that tunes these parameters automatically according to the *a posteriori* error estimate. Finally, it is shown how the method works in practice by presenting a numerical example.

Keywords: Finite element method; A posteriori error estimation; Variational multiscale method; Elliptic problem; adaptivity; Periodic coefficient

The focus of this paper is to present the adaptive variational multiscale method and show how it can be used to solve multiscale problems in an adaptive fashion. It is started by introducing a simple model problem.

1. The model problem

The Poisson equation is studied with a coefficient a and homogeneous Dirichlet boundary conditions: find $u \in H_0^1(\Omega)$ such that:

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega \tag{1}$$

Where Ω is a polygonal domain in $\mathbf{R}^d$, $d = 1, 2,$ or 3 with boundary Γ, $f \in L^2(\Omega)$, and $a \in L^\infty(\Omega)$ such that $a(x) > 0$ for all $x \in \Omega$. The variational form of Eq. (1) reads: find $u \in \mathcal{V} = H_0^1(\Omega)$ such that:

$$a(u, v) = (f, v) \quad \text{for all } v \in \mathcal{V} \tag{2}$$

with the bilinear form:

$$a(u, v) = (a\nabla u, \nabla v) \tag{3}$$

for all $u, v \in \mathcal{V}$. We mainly focus on multiscale phenomena arising from the coefficient a in Eq. (1).

2. The variational multiscale method

An important framework for solving multiscale problems is the variational multiscale method (VMM), see Hughes et al. [2,3]. The idea is to decompose the solution into fine $u_f \in \mathcal{V}_f$ and coarse $u_c \in \mathcal{V}_c$ scale contributions as in Eq. (4):

$$\begin{aligned} &a(u_c, v_c) + a(u_f, v_c) = (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ &a(u_f, v_f) = (f, v_f) - a(u_c, v_f) =: (R(u_c), v_f) \\ &\quad \text{for all } v_f \in \mathcal{V}_f \end{aligned} \tag{4}$$

The fine scale equations are solved in terms of the coarse scale residual $R(u_c)$, and finally we eliminate the fine scale solution from the coarse scale equation. This procedure leads to the modified coarse scale Eq. (5) where the modification accounts for the effect of fine scale behavior on the coarse scales.

$$a(u_c, v_c) + a(\mathcal{T}R(u_c), v_c) = (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c \tag{5}$$

Here $\mathcal{T}$ represents an approximate solution operator of the fine scale problem. In several works various ways of analytical modeling of $\mathcal{T}$ are investigated often based on bubbles or element Green's functions, see Hughes [2].

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3. Approximation of fine scales based on localized problems

In the adaptive variational multiscale method (AVMM), see Larson et al. [4,5,6] the fine scale equations of Eq. (4) are decoupled by a partition of unity and solved numerically on patches.

We let $\mathcal{N}$ be the set of coarse nodes and $\mathcal{V}_c$ be the finite element space of continuous piecewise linears polynomials on the coarse mesh. We let $u_f = \Sigma_{i\in\mathcal{N}} u_{f,i}$ where:

$$a(u_{f,i},\ v_f) = (\varphi_i R(u_c),\ v_f) \quad \text{for all } v_f \in \mathcal{V}_f \tag{5}$$

and $\{\varphi_i\} i\in\mathcal{N}$ *is a partition of unity, e.g. the set of Lagrange basis functions in* $\mathcal{V}_c$, be the solution to the decoupled fine scale equations.

We introduce this expansion of u_f in the right-hand side of the fine scale equation (4) and get: find $u_c \in \mathcal{V}_c$ and $u_f = \Sigma_{i\in N} u_{f,i} \in \mathcal{V}_f$ such that:

$$a(u_c,\ v_c) + a(u_f,\ v_c) = (f,\ v_c) \quad \text{for all } v_c \in \mathcal{V}_c,$$
$$a(u_{f,i},\ v_f) = (\varphi_i R(u_c),\ v_f) \quad \text{for all } v_f \in \mathcal{V}_f \text{ and} i \in \mathcal{N} \tag{6}$$

The next step is to solve the fine scale equations approximately. For each element in the partition of unity we associate a domain ω_i on which we solve Dirichlet problems. We often use coarse mesh stars of many layers as local domains. By adding a layer we mean adding all coarse elements bordering the star. The local domain ω_i contains the support of the element in the partition of unity and is large enough to give a good approximate solution. The quality of the solution is controlled by error estimates. We now define the local finite element space $\mathcal{V}_f^h(\omega_i)$ associated with node i. We refine the coarse mesh on the patch ω_i and let $\mathcal{V}_f^h(\omega_i)$ be the fine part of the hierarchical basis on this mesh.

The resulting method reads: find $U_c \in \mathcal{V}_c$ and $U_f = \Sigma_i^n U_{f,i}$ where $U_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that:

$$a(U_c,\ v_c) + a(U_f,\ v_c) = (f,\ v_c) \quad \text{for all } v_c \in \mathcal{V}_c,$$
$$a(U_{f,i},\ v_f) = (\varphi_i R(U_c),\ v_f) \quad \text{for all} v_f \in \mathcal{V}_f^h(\omega_i) \text{ and } i \in \mathcal{N} \tag{7}$$

Since the functions in the local finite element spaces $\mathcal{V}_f^h(\omega_i)$ are equal to zero on $\partial\omega_i$, U_f and therefore U will be continuous. If we just have fine scale features on part of the domain we only solve local problems for these areas. We denote coarse nodes in these areas $\mathcal{F}$ and the rest $\mathcal{C}$. If we write the method in matrix form we would get:

$$(A + T)\, U_c = b - d \tag{8}$$

where A and b are the standard finite element stiffness matrix and load vector and the T matrix and d vector arises in analogy with Eq. (5) since $\mathcal{T}\ (R(U_c))$ is affine in U_c. To get an idea of how the localized solution $U_{f,i}$ behaves when the domain ω_i increases we plot different

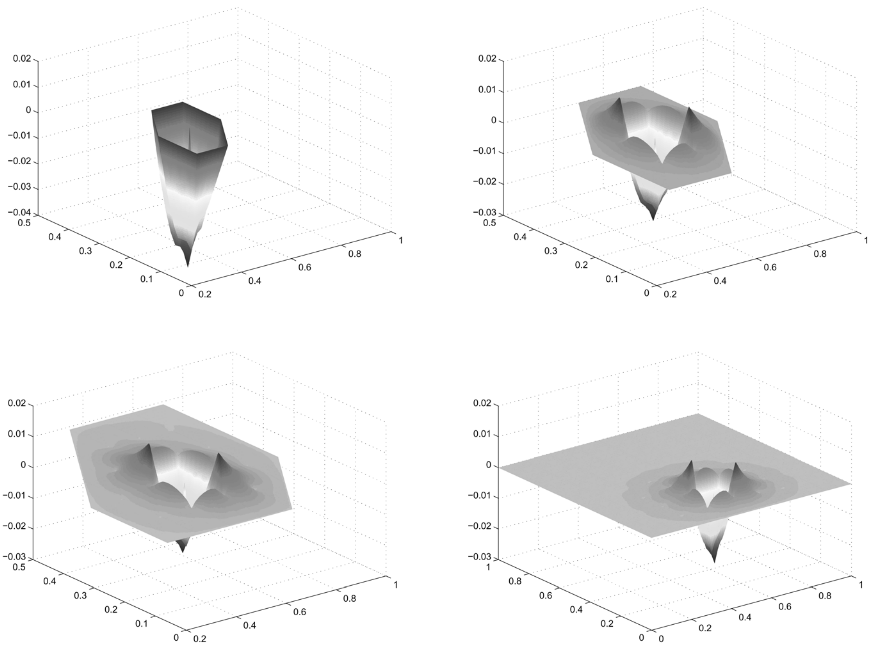

Fig. 1. A typical localized solution $U_{f,i}$ of the fine scale equations in a smooth region using one-, two-, and three-layer stars, and the entire domain.

solutions $U_{f,i}$ in Fig. 1. Since $U_{f,i}$ is solved in the slice space $\mathcal{V}_f$ and since the right-hand side of the fine scale equations of (7) has the same support as φ_i, $U_{f,i}$ will decay rapidly towards the boundary of ω_i, this can also be seen in Fig. 1. We can see that one-layer stars appears to give bad accuracy while two- and more layer stars capture the features of the correct solution.

4. Error estimation

In Larson et al. [6] we present the following a posteriori error estimate for the adaptive variational multiscale method in the energy norm $\| e \|_a^2 = a(e,e)$.

$$\| e \|_a^2 \leq C \sum_{i \in C} \| H\mathcal{R}(U_c) \|_{\omega_i}^2 \ \| \frac{1}{\sqrt{a}} \|_{L^\infty(\omega_i)}^2 + C \sum_{i \in \mathcal{F}} \left(\| \sqrt{H}\Sigma\,(U_{f,i}) \|_{\partial\omega_i \setminus \Gamma}^2 + \| h\mathcal{R}_i\,(U_{f,i}) \|_{\omega_i}^2 \right) \ \| \frac{1}{\sqrt{a}} \|_{L^\infty(\omega_i)}^2 \tag{9}$$

where

$$(-\Sigma(U_{f,i}),\ v_f)_{\partial\omega_i} = (\varphi_i \mathcal{R}(U_c),\ v_f)_{\omega_i} - a(U_{f,i},\ v_f)_{\omega_i}, \quad \text{for all } v_f \in V_f^h(\bar{\omega}_i) \tag{10}$$

Here $\mathcal{R}(U_c)$ and $\mathcal{R}_i(U_{f,i})$ are bounds of the coarse and fine scale residual and $\Sigma(U_{f,i})$ is a variational approximation of $\partial_n U_{f,i}$ on $\partial\omega_i$. The contributions to the error can easily be understood. If no fine scale equations are solved we obtain the first term in the estimate; the first part of the second sum measures the effect of restriction to patches; and finally the second part measures the influence of the fine scale mesh parameter h.

For the case of periodic oscillations in $a = a(x/\epsilon)$ we get:

$$\| e \|_a^2 \leq C \left(\frac{h}{\varepsilon}\right)^2 \| f \|^2 + C \sum_{i \in \mathcal{N}} \| \sqrt{H}\Sigma\,(U_{f,i}) \|_{\partial\omega_i \setminus \Gamma}^2 \| \frac{1}{\sqrt{a}} \|_{L^\infty(\omega_i)}^2 \tag{11}$$

Here local problems are solved for all nodes since all areas are equally hard to resolve. Again we see clearly that $\|\Sigma(U_{f,i})\|_{\partial\omega_i}$ which depends on the number of layers, and the fine scale mesh size h needs to be balanced. The coefficient a is periodic so we just need to solve a few localized problems since the correction matrix for the coarse scale computations will be identical for most patches.

In Larson et al. [4] present an error estimate is presented of the adaptive variational multiscale method for a linear function of the error.

5. Adaptive algorithm

A simple adaptive algorithm is presented, based on the error estimate in Eq. (11).

1. Give starting values for the refinement level r where $h = H/2^r$ and the number of layers L of the extended stars ω_i.
2. Solve Eq. (6) to get U_c.
3. Calculate $R_i = \left(\frac{h}{\varepsilon}\right)^2 \| \varphi_i^{1/2} f \|^2$ and $L_i = \| \sqrt{H}\Sigma\,(U_{f,i}) \|_{\partial\omega_i \setminus \Gamma}^2 \| \frac{1}{\sqrt{a}} \|_{L^\infty(\omega_i)}^2$ for each coarse node i.
4. If the levels of R_i and L_i are acceptable stop or else refine the fine scale mesh if $R_i > L_i$ or increase the fine scale domain size if $R_i < L_i$ and return to 2.

6. Numerical examples

We let Ω be the unit square and we let the coefficient a oscillate rapidly with period H according to Fig. 2. Since we have a periodic coefficient we use a constant h and L for all local problems and use the fact that many of them give equivalent contributions to the total modified stiffness matrix. In this way, a simple implementation of the method in Matlab can still handle very fine oscillations ϵ. The limit is the size of the coarse scale calculation.

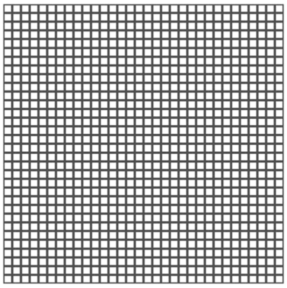

Fig. 2. The coefficient is discontinuous with the values $a = 1$ on the white areas and $a = 0.05$ on the dark areas. The figure is an enlargement of a small part of the domain.

Let $f = 1$, $H = 1/128$, and start the adaptive algorithm with $r = L = 1$. In Fig. 3 it is shown how the error indicators R_i and L_i change through the iterations. As seen in Fig. 3, the algorithm first performs two refinements to resolve the lattice of with $H/8$. Then one layer is added to the stars and then one more refinement and so on. It appears to be simple to adjust the layers so that the main contribution to the error is the fine scale mesh size. This is possible since the indicator L_i drops quickly while increasing the number of layers.

As mentioned before, calculating a modified stiffness matrix rather than using an iterative approach is very

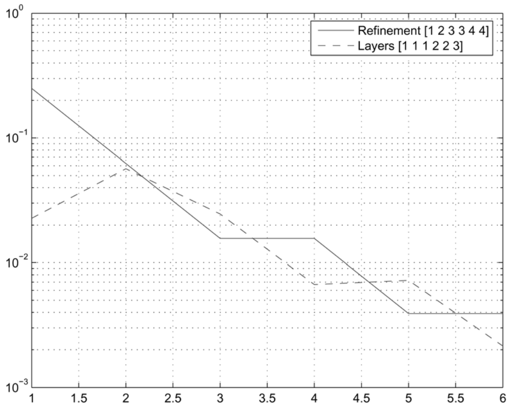

Fig. 3. The error indicators during six iterations in the adaptive algorithm.

efficient in the periodic setting. To understand the method it is interesting to know how the method actually modifies the stiffness matrix. This is done by studying the spectrum of the resulting matrix $A + T$, see Eq. (8), for different number of layers in Fig. 4. We study the 20 lowest and most significant eigenvalues. The first thing we note is that the eigenvalues of $A + T$ always are smaller than the ones of A. This is natural since the discretization increases eigenvalues of the operator. It is also seen that after two layers a very nice agreement with the correct spectrum we like to approximate.

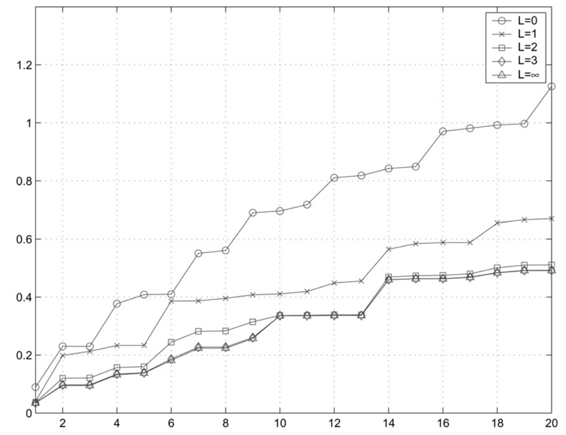

Fig. 4. The 20 lowest eigenvalues of the matrix $A + T$ for fine scale problems solved using no stars, one-, two-, and three-layer stars, and the entire domain.

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