Reliable finite elements for the analysis of piezoelectric shells

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Abstract

This paper presents a reliable and robust low-order finite shell element without zero energy modes or locking effects for smart structures applications, especially for layered plate and shallow shell structures with adhered piezoelectric patches serving as actuators or sensors. Four-noded finite elements are developed on the basis of the assumed natural strain (ANS) method taking into account piezoelectric material behavior. The ANS technique adapted from purely mechanical finite element formulations is combined with a two-field and a three-field variational formulation of the electromechanical problem. In contrast to the two-field variational formulation, in which displacements and electric potentials serve as independent variables, the three-field variational formulation also takes the dielectric displacement into consideration. Numerical examples are presented in which various finite elements are compared with each other.

Keywords: Smart structures; Finite elements; Piezoelectricity; ANS method; Shallow shell; Coupled field problems

1. Introduction

In adaptive structures the implementation of piezoelectric material is gaining in importance for actuation as well as sensing applications. Therefore, well established design and analysis tools such as the finite element method have to be extended in order to enable the further development of adaptive structures. However, despite the increasing demand for efficient and particularly reliable low-order element formulations that are applicable for a large variety of actuation and sensing problems, only a few plate and shell elements meet these requirements so far. Although the techniques for locking-free and stable finite elements are well known for purely mechanical formulations [1], this knowledge has still to find its way into coupled electromechanical problems. In this paper the ANS method is integrated into a four-noded shell element that allows for the analysis of shallow shells with piezoelectric patches bonded to the surfaces. The evolved finite element formulation, which is based on the first-order shear deformation theory, is locking-free and has no zero energy modes.

Another possibility to avoid locking phenomena in purely mechanical analyses or to enable a simple implementation of nonlinear constitutive equations is the use of hybrid-mixed methods, which have been used successfully in the past [2]. So additionally to the ANS method, an analogous hybrid finite element formulation will be presented in this paper for electromechanically coupled problems in order to allow for the later use of nonlinear material models. First investigations concerning the material nonlinearity have been done in Ghandi et al. [3] and Lammering et al. [4]. A variational principle for electromechanical systems is derived augmenting the total potential energy in the Hu-Washizu functional further. This is achieved by integrating the electric field–electric potential relation into the formulation via a Lagrange multiplier. Based on this variational principle, electromechanically coupled hybrid finite elements are obtained in which the dielectric displacement serves as an additional independent variable and can be condensed on the element level [3,5].

The performance of these different finite element formulations is tested in numerical examples in which the piezoelectric material works as actuator or as sensor. It will be shown that the numerical results agree well with experiments and with examples from the literature for static and dynamic applications.

2. Constitutive equations

The widely used material equations are developed with the electric enthalpy density H, where the strains

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$\mathbf{E}^{\mathrm{m}}$ and the electric field $\mathbf{E}^{\mathrm{el}}$ are chosen as the independent field variables. Considering H as

$$H = \frac{1}{2}(\mathbf{C} : \mathbf{E}^{\mathrm{m}}) : \mathbf{E}^{\mathrm{m}} - \mathbf{e} \cdot \mathbf{E}^{\mathrm{el}} : \mathbf{E}^{\mathrm{m}} - \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{E}^{\mathrm{el}} \cdot \mathbf{E}^{\mathrm{el}} \tag{1}$$

the linear constitutive relations for piezoelectric materials can be deduced from partial derivatives of the electric enthalpy with respect to $\mathbf{E}^{\mathrm{m}}$ and $\mathbf{E}^{\mathrm{el}}$ [6], yielding the stress tensor $\boldsymbol{\sigma}$ and the dielectric displacement $\mathbf{D}$:

$$\boldsymbol{\sigma} = \frac{\partial H}{\partial \mathbf{E}^{\mathrm{m}}} \quad \mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}^{\mathrm{el}}} \tag{2}$$

Therefore, one obtains

$$\boldsymbol{\sigma} = \mathbf{C} : \mathbf{E}^{\mathrm{m}} - \mathbf{e} \cdot \mathbf{E}^{\mathrm{el}} \tag{3a}$$

$$\mathbf{D} = \mathbf{e}^{\mathrm{T}} : \mathbf{E}^{\mathrm{m}} + \boldsymbol{\varepsilon} \cdot \mathbf{E}^{\mathrm{el}} \tag{3b}$$

where Eqs. (3a) and (3b) describe the coupling between the mechanical and electrical material behavior. The fourth-order elasticity tensor $\mathbf{C}$ is used in the same way as in Hooke's law and its material constants are determined at a constant electric field $\mathbf{E}^{\mathrm{el}}$. The electrical behavior is formulated through the second-order permittivity, ε, which is measured at constant strain $\mathbf{E}^{\mathrm{m}}$. The electromechanical coupling is realized through the third-order piezoelectric modulus $\mathbf{e}$. These coupled constitutive equations have to be adapted for thin shell structures to the plane stress state, cf. Lammering et al. [7]. The coupling condition for the material parameters is a necessary and sufficient condition for the enthalpy function

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{E}^{\mathrm{el}}} = -\frac{\partial \mathbf{D}}{\partial \mathbf{E}^{\mathrm{m}}} \tag{4}$$

For the three-field variational formulation a different set of independent variables is needed, see Section 4. Therefore, a conjugate form of the constitutive relations is derived where $\mathbf{E}^{\mathrm{m}}$ and $\mathbf{D}$ are chosen as the independent quantities. The associated constitutive equations can be deduced from the internal energy density U [8]. Through a Legendre-transformation, the electric enthalpy density H of Eq. (1) can be transferred to the chosen potential, which is needed for the three-field formulation, so that the conjugate constitutive relations read

$$\boldsymbol{\sigma} = \mathbf{C}^{\mathrm{d}} : \mathbf{E}^{\mathrm{m}} - \mathbf{h} \cdot \mathbf{D} \tag{5a}$$

$$\mathbf{E}^{\mathrm{el}} = -\mathbf{h}^{\mathrm{T}} : \mathbf{E}^{\mathrm{m}} + \boldsymbol{\beta} \cdot \mathbf{D} \tag{5b}$$

where $\mathbf{C}^{\mathrm{d}}$ denotes the elasticity tensor measured at constant dielectric displacement $\mathbf{D}$, $\boldsymbol{\beta}$ is the impermittivity tensor deduced at constant mechanical strain $\mathbf{E}^{\mathrm{m}}$ and $\mathbf{h}$ is the piezoelectric modulus. As before, a reduction to the plane stress state has to be performed for thin shell applications.

3. Two-field weak form and elements

Starting from the well known balance of momentum and charge conservation equations, one obtains after multiplication with the virtual displacement $\delta\mathbf{u}$ and the virtual electric potential $\delta\Phi$, integration over the volume of the body $\mathcal{B}$, and some straightforward calculations, the weak form of equilibrium of the coupled electromechanical problem [5]:

$$\begin{aligned}\delta\mathcal{G}(\mathbf{u},\ \delta\mathbf{u},\ \Phi,\ \delta\Phi) = \int_{\mathcal{B}} [\delta\mathbf{E}^{\mathrm{m}} : \boldsymbol{\sigma} + \delta\mathbf{E}^{\mathrm{el}} \cdot \mathbf{D}]\, \mathrm{dV}\ + \\ \int_{\mathcal{B}} \rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \delta\mathbf{u}\, \mathrm{dV} - \int_{\partial\mathcal{B}_\sigma} \delta\mathbf{u} \cdot \bar{\mathbf{t}}\, \mathrm{dA}\ - \\ \int_{\mathcal{B}} \delta\mathbf{u} \cdot \rho_o \mathbf{b}\, \mathrm{dV} - \int_{\partial\mathcal{B}_D} \delta\Phi \bar{d}\, \mathrm{d}A = 0\end{aligned} \tag{6}$$

where $\bar{\mathbf{t}}$ denotes the traction vector on the surface $\partial\mathcal{B}_\sigma$, $\bar{\mathbf{d}}$ is the charge density on the surface $\partial\mathcal{B}_D$, $\rho_o\mathbf{b}$ is the specific body force and $\rho_o\partial^2\mathbf{u}/\partial t^2$ the specific inertia force. Using Eqs. (3a) and (3b) one obtains

$$\begin{aligned}\int_{\mathcal{B}} \delta\mathbf{E}^{\mathrm{m}} : (\mathbf{C} : \mathbf{E}^{\mathrm{m}} - \mathbf{e} \cdot \mathbf{E}^{\mathrm{el}})\, dV + \int_{\mathcal{B}} \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \delta\mathbf{u}\, dV - \\ \int_{\partial\mathcal{B}_\sigma} \delta\mathbf{u} \cdot \bar{\mathbf{t}}\, dA - \int_{\mathcal{B}} \delta\mathbf{u} \cdot \rho_0 \bar{\mathbf{b}}\, dV = 0 \\ \int_{\mathcal{B}} \delta\mathbf{E}^{\mathrm{el}} \cdot (-\mathbf{e}^{\mathrm{T}} : \mathbf{E}^{\mathrm{m}} - \boldsymbol{\varepsilon} \cdot \mathbf{E}^{\mathrm{el}})\, dV - \int_{\partial\mathcal{B}_D} \delta\Phi \bar{d}\, dA = 0\end{aligned} \tag{7}$$

Subsequently, the stress resultants are computed through integration of the respective stresses over the thickness t_i of each layer and summation over all layers n_{i}.

A symmetric cross section is assumed so that the bending moments do not contribute to the normal forces. To develop an isoparametric four-noded shallow shell element, the displacements $\mathbf{u}$, the rotations $\boldsymbol{\psi}$ and the electric potential $\boldsymbol{\Phi}$ are approximated by bilinear shape functions in the inplane direction of the shell. The developed finite elements will suffer from transversal shear locking. This is overcome with the concept of the assumed natural strain (ANS) method, which is a well-known technique for pure mechanical plate and shell elements. Although several concepts are well established for pure mechanical analysis, these methods are rarely used for electromechanical structures [9].

The necessary condition of the ANS method is that the shear strains are decoupled from the remaining strains, which is also fulfilled in the case of coupled electro-mechanical problems. An advantage of this technique is that only a modified matrix $\bar{\mathbf{B}}_{\mathrm{S}}$ for the shear terms is developed but that the structure of the system of equations remains unchanged.

Additionally, a quadratic distribution of the electric potential across the piezoelectric layers is taken into consideration. This distribution preserves the charge conservation condition in thickness direction, cf. Lammering et al. [7]. As a result, additional terms appear in the stiffness matrix that are not present in the mostly assumed case of a linear distribution. The respective equations are not shown here for the sake of brevity.

4. Multi-field weak forms and elements

In purely mechanical problems, multi-field weak forms are used for the construction of elements that do not lock and possess other performance enhancements. A weak form for the electromechanical problem, which results in a finite element with displacement, rotation, electric potential, electric field and dielectric displacement degrees of freedom, is obtained by the following variational principle:

$$\delta\mathcal{G}^*(\mathbf{u},\ \delta\mathbf{u},\ \boldsymbol{\Phi},\ \delta\boldsymbol{\Phi},\ \mathbf{E}^{\mathrm{el}},\ \delta\mathbf{E}^{\mathrm{el}},\ \mathbf{D},\ \delta\mathbf{D}) = \delta\mathcal{G}(\mathbf{u},\ \delta\mathbf{u},\ \boldsymbol{\Phi},\ \delta\boldsymbol{\Phi}) + \int_{\mathcal{B}} \delta\mathbf{D}\cdot(\mathbf{E}^{\mathrm{el}} + \mathrm{Grad}\ \Phi)\ dV + \int_{\mathcal{B}} \mathbf{D}\cdot(\delta\mathbf{E}^{\mathrm{el}} + \mathrm{Grad}\ \delta\,\Phi)\ dV = 0 \tag{8}$$

where $\delta\mathbf{D}$ denotes the virtual dielectric displacement and $\delta\mathbf{E}^{\mathrm{el}}$ the virtual electric field. This electric field–electric potential relation has been introduced into the variational principle as a constraint by a Lagrange multiplier that is identified as the dielectric displacement. After some straightforward calculations, it is apparent that the virtual electric field is dropped out of the formulation. So the only dependent electrical quantity is now the electric field $\mathbf{E}^{\mathrm{el}}$, which has to be computed from the constitutive equation (5b) and not as in the two-field formulation from the electric potential by gradient analysis given by $\mathbf{E}^{\mathrm{el}} = -\mathrm{Grad}\ \Phi$. The variational principle has now the following form:

$$\int_{\mathcal{B}} \boldsymbol{\sigma}{:}\delta\mathbf{E}^{\mathrm{m}} dV + \int_{\mathcal{B}} \rho_0 \frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\ dV + \int_{\mathcal{B}} \delta\mathbf{D}\cdot(\mathbf{E}^{\mathrm{el}} + \mathrm{Grad}\Phi)\ dV + \mathbf{D}\cdot\mathrm{Grad}\ \delta\,\Phi\ dV = \int_{\partial\mathcal{B}_\sigma} \bar{\mathbf{t}}\cdot\delta\mathbf{u}\ dA + \int_{\mathcal{B}} \rho_0\mathbf{b}\ \cdot\delta\mathbf{u}\ dV + \int_{\partial\mathcal{B}_D} \bar{d}\delta\,\Phi\ dA \tag{9}$$

This formulation allows for the implementation of the constitutive equations (5a) and (5b), yielding the following three-field variational formulation:

$$\begin{aligned}
&\int_{\mathcal{B}} \delta\mathbf{E}^{\mathrm{m}} : (\mathbf{C}_{\mathrm{d}} : \mathbf{E}^{\mathrm{m}} - \mathbf{h}\cdot\mathbf{D})\ dV + \int_{\mathcal{B}} \rho_0 \frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\ dV - \int_{\partial\mathcal{B}_\sigma} \delta\mathbf{u}\cdot\bar{\mathbf{t}}\ dA - \int_{\mathcal{B}} \delta\mathbf{u}\cdot\rho_0\bar{\mathbf{b}}\ dV = 0\\
&\int_{\mathcal{B}} \mathrm{Grad}\delta\ \Phi\cdot\mathbf{D}\ dV - \int_{\partial\mathcal{B}_D} \delta\ \Phi\bar{d}\ dA = 0\\
&\int_{\mathcal{B}} \delta\mathbf{D}\cdot(-\mathbf{h}^{\mathrm{T}} : \mathbf{E}^{\mathrm{m}} + \boldsymbol{\beta}\cdot\mathbf{D} + \mathrm{Grad}\ \Phi)\ dV = 0
\end{aligned} \tag{10}$$

Note that for a locking-free formulation, the shear terms also have to be calculated with the modified operator matrix $\bar{\mathbf{B}}_{\mathrm{s}}$.

5. Example

A square plate that is pin supported at its corner nodes serves as a first numerical example. The geometric and material properties are taken from Lammering et al. [7]. Here the electric potential that is needed to regain the undeformed shape of the system is investigated. The results of the numerical calculations are depicted in Fig. 1, where the deflection in the axes of symmetry is shown. Starting from 0 V the voltage has to be increased to a maximum voltage of 75 V to averagely obtain the initial undeformed state. Both formulations with the ANS method agree well with each other whereas elements with selective reduced integration would suffer from zero energy modes. Figure 2 shows the deformed shape at the maximum electric potential of 75 V. It can be seen that the electrical load counteracts the mechanical load leading to a displacement field that vanishes averagely but not exactly at each point of the plate. In order to achieve an overall zero displacement, optimization tools have to be used for computation of the appropriate actuator shape. Further examples are concerned with dynamic applications as well as with piezoelectric material in sensor function. These examples are not shown here for the sake of brevity.

6. Conclusion

Coupled electromechanical problems and their analysis using the finite element method were in the focus of this paper. The well-known two-field formulation in which the displacements and the electric potential serve as unknowns has been enhanced with the ANS method, so that a reliable two-field finite element formulation has been developed. This approach has also been transferred to a three-field formulation, which is helpful when the

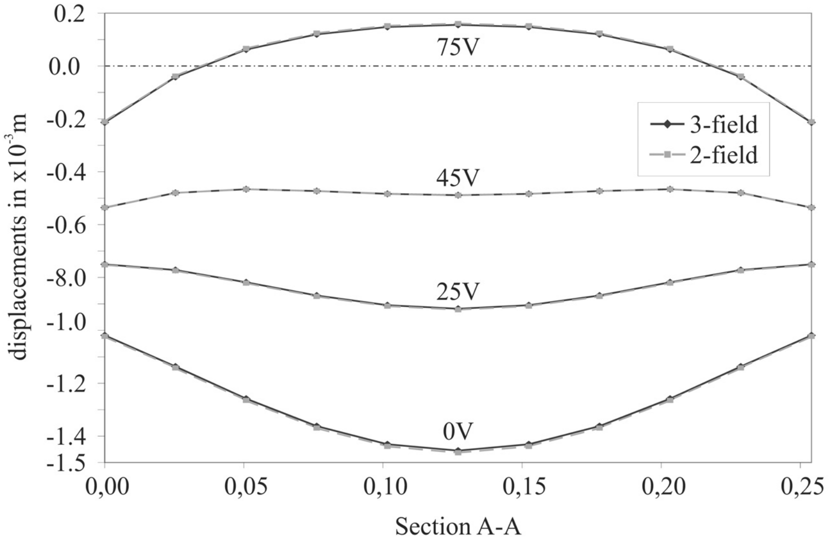

Fig. 1. Deflection in the axes of symmetry of the pin-supported plate at various electric potentials for a uniformly distributed mechanical load.

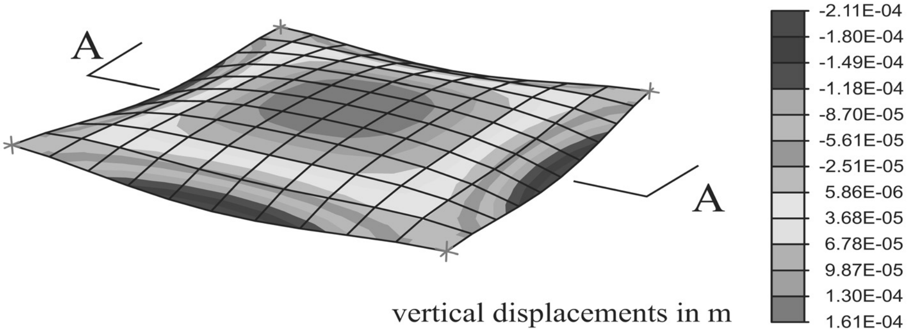

Fig. 2. Deformed shape of the pin-supported plate at the maximum electric potential of 75 V.

material behavior cannot be described through linear approximations properly, e.g. for high signal ranges. Then multi-field formulations and related elements are an efficient alternative for these systems.

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