An efficient numerical modelling of anisotropic structural behaviour in large strain elastoplasticity

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Abstract

The paper deals with aspects of anisotropic elastoplastic analysis at finite strain. The material model is based on the multiplicative decomposition of the deformation gradient. An anisotropic elastic constitutive law, described in invariant setting by the use of structural tensors, is presented. An anisotropic Hill-type yield criterion, described by an Eshelby-like stress tensor and further structural tensors, is developed. It considers nonlinear isotropic hardening as well. Explicit results for the specific case of orthotropic anisotropy are given. An accurate and trivial wise objective integration algorithm employing the exponential map is given. The numerical example demonstrates the influence of anisotropy on the elastoplastic deformation process as well as the robustness and accuracy of the proposed formulation.

Keywords: Elastic orthotropy; Orthotropic yield function; Multiplicative inelasticity; Finite strain; Isotropic hardening; Exponential map

1. Introduction

In recent years, the topic of anisotropic inelasticity at finite strains has been attracting considerable attention because of its relevance to the deformation of structural components and the many still open questions encountered in its modelling and computation. Anisotropic material responses arise from the micro structure of the material and it can be described by different constitutive laws. Generally, constitutive laws can be divided into two classes: those related to the thermodynamical forces, where the constitutive law can be defined through a thermodynamical potential; and those related to the evolution of the internal variables. Perhaps the most popular example of an anisotropic constitutive law is Hill's yield criterium [1]. In the small strain regime, inelasticity is based on the additive decomposition of the strain tensor and, accordingly, the formulation of the anisotropy does not cause significant difficulties, de Borst et al. [2]. Such additive decompositions can be extended to the large strain regime resulting in a rather simple formulation of anisotropy as shown by Papadopoulos et al. [3] and Schroeder et al. [4]. However, successful additive decompositions can be expected only if the logarithmic strain measure is employed.

The multiplicative decomposition of the deformation gradient may be faced with a variety of problems. The work of Sansour et al. [5] deals with the implications of the multiplicative decomposition of the deformation gradient within an anisotropic elastic constitutive law for viscoplastic materials. The same decomposition was also employed in an anisotropic formulation of elastoplasticity at large strains presented by Eidel et al. [6]. However, the authors made certain simplifications in the definition of the orthotropic yield function and employed a numerically derived tangent modulus leaving issues related to the consistent tangent modulus not addressed.

This contribution presents a theoretical model of anisotropic inelasticity and a corresponding numerical implementation where the multiplicative decomposition of deformation gradient is employed. Anisotropy is encountered in the elastic constitutive law as well as in the yield function. Specifically Hill's anisotropic yield function is modified so as to fit into the material frame of the theoretical formulation. Following the work of Sansour et al. [7] the anisotropic form of the free energy function is achieved using a set of invariants to depend on the so-called structural tensors and a right Cauchy–

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Green type tensor. The structural tensors represent the privileged material directions which are not altered during the deformation process. The anisotropic yielding response is described using a quadratic form of the yield function to depend on a certain set of invariants. These invariants are functions of a deviatoric material Eshelby-like stress tensor and the structural tensors. Explicit results are given for the specific case of orthotropic anisotropy, where a nonlinear isotropic hardening response is also considered. The efficiency of the proposed algorithms are demonstrated by a numerical example.

2. Constitutive relations and numerical formulation

For the description of inelastic deformation, the well established multiplicative decomposition of the deformation gradient in an elastic part, $\mathbf{F_e}$, and in an inelastic part, $\mathbf{F_p}$, is assumed:

$$\mathbf{F} = \mathbf{F_e}\mathbf{F_p} \tag{1}$$

On the basis of the above decomposition, the following deformation tensors of the Cauchy–Green type are defined:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{C_e} = \mathbf{F_e}^{\mathrm{T}}\mathbf{F_e}, \quad \mathbf{C_p} = \mathbf{F_p}^{\mathrm{T}}\mathbf{F_p} \tag{2}$$

where $\mathbf{C}$ and $\mathbf{C_p}$ constitute material tensors, and $\mathbf{C_e}$ is

Table 1
Constitutive equations

Dissipation inequality:

$$D = \Xi : \mathbf{L} - \rho\left(\frac{\partial\psi}{\partial\mathbf{C_e}}\dot{\mathbf{C}}_\mathbf{e} - \frac{\partial\psi}{\partial Z}\dot{Z}\right) \geq 0 \tag{3}$$

Decomposition of the free energy:

$$\psi = \psi_e + \psi_p \tag{4}$$

Elastic free energy:

$$\psi_e = \sum_{i=1}^{3}\left[\alpha_i J_i + \alpha_{(i+3)}J_i^2 + \alpha_{(i+9)}J_{(i+3)} + \frac{1}{2}\sum_{j=1}^{3}\alpha_{(i+j+4)}J_iJ_j\right], \quad \text{for } i \neq j \tag{5}$$

Elastic anisotropy invariants:

$$J_i = \mathrm{tr}[({}_i\mathbf{M_e})\mathbf{C_e}], \qquad J_{(i+3)} = \mathrm{tr}\left[({}_i\mathbf{M_e})\mathbf{C_e}^2\right], \quad i = 1,2,3 \tag{6}$$

Structural tensors at anisotropic elastic constitutive law:

$${}_i\mathbf{M_e} = {}_i\mathbf{v_e} \otimes {}_i\mathbf{v_e}, \qquad i = 1,2,3 \tag{7}$$

where ${}_i\mathbf{v_e}$ are privileged material directions.

Reduced dissipation inequality:

$$D_r = \Xi : \mathbf{L_p} + Y \cdot \dot{Z} \geq 0 \tag{8}$$

Eshelby's stress tensor:

$$\Xi = 2\rho\mathbf{C}\mathbf{F_p}^{-1}\frac{\partial\psi}{\partial\mathbf{C_e}}\mathbf{F_p}^{-\mathrm{T}}, \tag{9}$$

$$\Xi = 2\rho\left\{\sum_{i=1}^{3}\left[\frac{\partial\psi}{\partial J_i}\mathbf{C}\mathbf{C_p}^{-1}({}_i\bar{\mathbf{M}}_\mathbf{e}) + \frac{\partial\psi}{\partial J_{(i+3)}}(\mathbf{C}\mathbf{C_p}^{-1}({}_i\bar{\mathbf{M}}_\mathbf{e})\mathbf{C}\mathbf{C_p}^{-1} + \mathbf{C}\mathbf{C_p}^{-1}\mathbf{C}\mathbf{C_p}^{-1}({}_i\bar{\mathbf{M}}_\mathbf{e}))\right]\right\} \tag{10}$$

Modified structural tensor:

$$({}_i\bar{\mathbf{M}}_\mathbf{e}) = \mathbf{F_p}^{\mathrm{T}}({}_i\mathbf{M_e})\mathbf{F_p}^{-\mathrm{T}}, \qquad i = 1,2,3 \tag{11}$$

Table 2
Evolution equations

Principle of maximum dissipation:

$$\int \left(-\left(\Xi : \mathbf{L_p} + Y \cdot \dot{Z}\right) + \lambda\phi(\Xi, Y)\right) ds = stat \tag{12}$$

Orthotropic yield function:

$$\phi = \sqrt{\frac{2}{3}}\sigma_{11}^{o}[\sqrt{\chi} - (1 - Y)] \tag{13}$$

where:

$$\chi = \sum_{i=1}^{3}\left[\beta_i I_i^2 + \beta_{(i+6)} I_{(i+3)} + \frac{1}{2}\sum_{j=1}^{3}\beta_{(i+j+1)} I_i I_j\right] \tag{14}$$

$$Y = -HZ - (\sigma_\infty - \sigma_{11}^{o}) \cdot (1 - \exp(-\eta Z)) \tag{15}$$

Yield function invariants:

$$I_i = \mathrm{tr}\left[\left({}_i\mathbf{M_y}\right)\mathrm{dev}\,\Xi\right], \qquad I_{(i+3)} = \mathrm{tr}\left[\left({}_i\mathbf{M_y}\right)(\mathrm{dev}\,\Xi)^2\right], \qquad i = 1, 2, 3 \tag{16}$$

Structural tensors at anisotropic yield function:

$${}_i\mathbf{M_y} = {}_i\mathbf{v_y} \otimes {}_i\mathbf{v_y}, \qquad i = 1, 2, 3 \tag{17}$$

where ${}_i\mathbf{v_y}$ are privileged material directions.
Evolution equations:

$$\mathbf{L_p} = \lambda\frac{\partial\phi}{\partial\Xi} \qquad \Rightarrow \qquad \mathbf{L_p} = \lambda\sqrt{\frac{2}{3}}\frac{\sigma_{11}^{o}}{2\sqrt{\chi}}\frac{\partial\chi}{\partial\mathrm{dev}\,\Xi}\frac{\partial\mathrm{dev}\,\Xi}{\partial\Xi}, \tag{18}$$

$$\dot{Z} = \lambda\frac{\partial\phi}{\partial Y} \qquad \Rightarrow \qquad \dot{Z} = \sqrt{\frac{2}{3}}\lambda, \tag{19}$$

where:

$$\begin{aligned}\frac{\partial\chi}{\partial\mathrm{dev}\,\Xi} = \sum_{i=3}^{3} [2\beta_i I_i({}_i\mathbf{M_y})^{\mathrm{T}} + \beta_{(i+6)}\ (({}_i\mathbf{M_y})\ \mathrm{dev}\,\Xi + ({}_i\mathbf{M_y})\ \mathrm{dev}\,\Xi) \\ + \frac{1}{2}\sum_{j=1}^{3}\beta_{(i+j+1)}\ (({}_i\mathbf{M_y})^{\mathrm{T}}\ I_j + I_i({}_j\mathbf{M_y})^{\mathrm{T}})\Big]\end{aligned} \tag{20}$$

given with respect to the so-called intermediate configuration.

The basic constitutive equations are presented in Tables 1 and 2. In Table 1 the anisotropic elastic constitutive law, described by the elastic free energy function ψ, is presented. It also includes an expression for the dissipation in equality defined in a material setting. Herein, Y represents isotropic hardening with the energy conjugate internal variable Z, $\mathbf{L}$ is the right rate of the deformation gradient, α_1–α_{12} are material constants, $\mathbf{L_p}$ is the right rate of the inelastic deformations and ρ is the density at the reference configuration.

The orthotropic yield function and evolution equations are presented in Table 2. Herein, σ_{11}^{o} and β_1–β_9 are the material constants, H is the linear isotropic hardening parameter, σ_∞ is the saturation yield stress, η is a constitutive parameter quantifying the rate at which the saturation yield stress is attained during loading, and λ denotes the plastic multiplier.

The integration of the presented evolution equations is performed using the well-known predictor-corrector computational strategy [8]. The exponential map is used for updating the inelastic part of the deformation gradient and it ensures the fulfillment of the incompressibility condition of the inelastic deformation. Accordingly, the plastic parts of the deformation gradient at time step t_{n+1} may be expressed as

$$\mathbf{F_p}^{-1}\Big|_{n+1} = \exp(-\Delta T\,\mathbf{L_p})\mathbf{F_p}^{-1}\Big|_{n} \tag{21}$$

By inserting the update relations of the state variables in the yield function, a nonlinear scalar equation for the plastic multiplier is obtained. It has to be solved by employing a local Newton's iterative solution procedure.

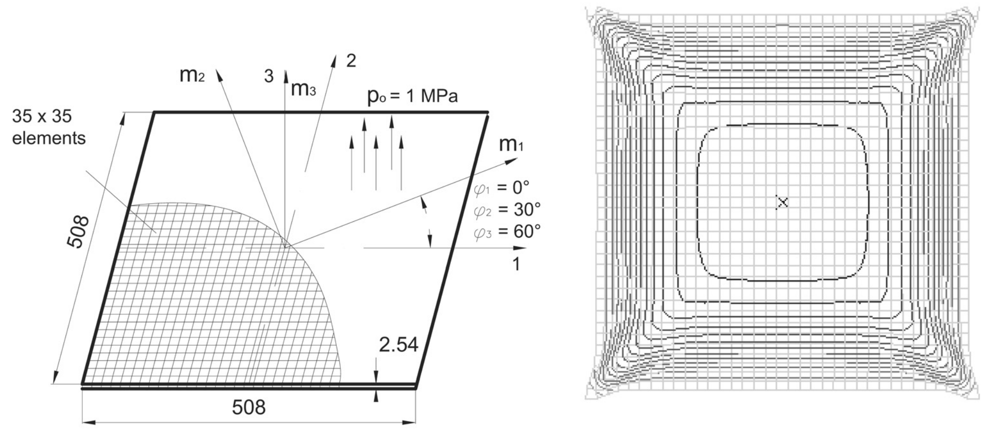
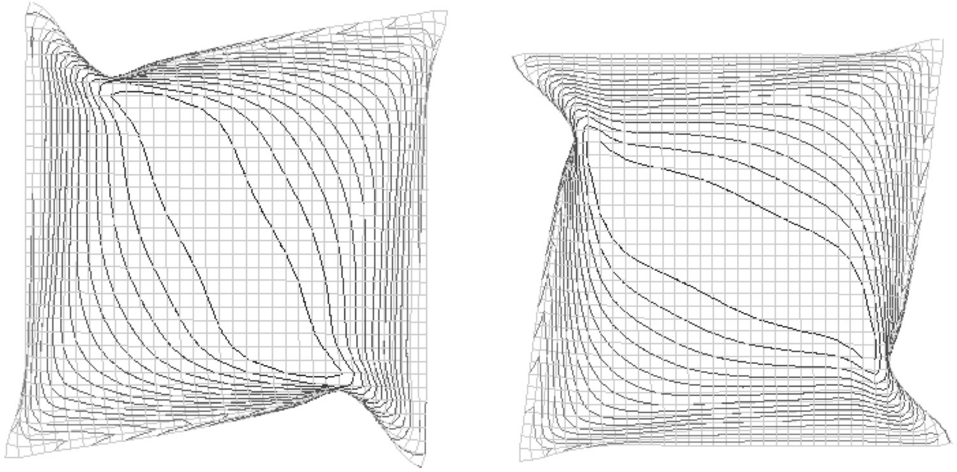


Fig. 1. Geometrical data and deformed configuration at angle of 0°.



(a) (b)

Fig. 2. Deformed configuration at angle of (a) 30°, (b) 60°.

In order to ensure quadratic convergence in the global iteration scheme an algorithmic tangent operator is derived. It is achieved by linearization of the second Piola–Kirchhoff tensor $\mathbf{S} = \mathbf{C}^{-1}\Xi$, with respect to the right Cauchy–Green deformation tensor $\mathbf{C}$

$$\frac{\partial \mathbf{S}}{\partial \mathbf{C}} = \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \Xi + \mathbf{C}^{-1} \frac{\partial \Xi}{\partial \mathbf{C}} \tag{22}$$

The theory and the computational algorithms have been implemented and applied to a shell finite element developed by Sansour et al. [8], allowing the use of complete three-dimensional constitutive laws.

3. Numerical example

As an example, deformation of a simply supported square plate, depicted in Fig. 1, is considered. The geometrical data are shown in the figure and the material data are Young's moduli $E_1 = 210\,\text{GPa}$, $E_2 = 84\,\text{GPa}$, $E_3 = 84\,\text{GPa}$, Poisson's ratios $\upsilon_{12} = 0.2285$, $\upsilon_{13} = 0.2426$, $\upsilon_{23} = 0.199$ and shear moduli are $G_{12} = 42\,\text{GPa}$, $G_{13} = 42\,\text{GPa}$, $G_{23} = 81\,\text{GPa}$. The initial yield stresses are $\sigma_{11} = 585\,\text{MPa}$, $\sigma_{22} = 810\,\text{MPa}$, $\sigma_{33} = 360\,\text{MPa}$, $\sigma_{12} = 286\,\text{MPa}$, $\sigma_{13} = 234\,\text{MPa}$, $\sigma_{23} = 260\,\text{MPa}$ and the isotropic hardening parameter $H = 2.0\,\text{GPa}$. The plate is subjected to the conservative load of $p_0 = 1\,\text{MPa}$ and whole plate is dicretized by 35×35 elements.

Figures 1 and 2 show deformed configurations of the plate with the rotation of the privileged directions of orthotropy relative to the fixed co-ordinate system given by the angles 0°, 30° and 60°. The isolines of the vertical displacement are plotted on these figures. A significant influence of the material privileged directions on the deformation process can be observed.

4. Conclusion

An efficient large strain elastoplastic material model for multiplicative inelasticity has been presented. The model employs an orthotropic constitutive law for the stress tensor and an orthotropic yield function in terms of material Eshelby-like stresses. An appropriate formulation of an orthotropic free energy function is considered. Nonlinear isotropic hardening is considered as well. The privileged directions of both the elastic constitutive law and the flow rule are described by structural tensors. Integration of the evolution equations is carried out using the exponential map and a consistent elastoplastic tangent modulus is derived. The numerical example demonstrates a significant influence of anisotropy and the privileged directions on the deformation process. The numerical algorithms are accurate and robust but it must be stressed, however, that the resulting algorithmic expressions are very involved.

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