A direct boundary integral method for viscoelastic-elastic composite materials

Y. Huang*, S.L. Crouch, S.G. Mogilevskaya

Department of Civil Engineering, University of Minnesota, Minneapolis, MN 55455, USA

Abstract

The paper is concerned with the problem of an infinite, isotropic Boltzmann viscoelastic plane containing a large number of randomly distributed, non-overlapping circular holes and perfectly bonded elastic inclusions. The holes and inclusions are of arbitrary size and the elastic properties of all of the inclusions can, in general, be different. The whole system is subjected to time-dependent stresses at infinity. The method of solution is based on a direct boundary integral approach for the problem of an infinite elastic plane containing multiple circular holes and elastic inclusions described by Crouch and Mogilevskaya [1], and a time-stepping strategy for general viscoelastic analysis described by Mesquita and Coda [2]. Numerical examples are included to demonstrate the accuracy and efficiency of the method.

Keywords: Composite material; Boundary integral method; Boltzmann model; Fourier series; Time stepping; Circular holes; Inclusions

1. Introduction

Composite materials often exhibit viscoelastic behavior. Thus, a robust and efficient numerical approach for modeling viscoelastic-elastic composite materials, especially subjected to varying loading condition, is desired.

In this paper, we combine the direct boundary integral equation approach for multiple holes and inclusions developed in [1] and the time-stepping approach for general viscoelastic analysis explained in [2], to consider the problem of an infinite, homogeneous, isotropic viscoelastic plane containing multiple circular holes and circular elastic inclusions, subjected to time-dependent loading at infinity (Fig. 1). The Boltzmann model is employed to simulate the viscoelastic plane.

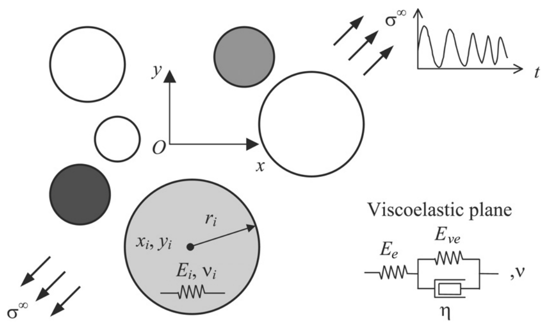

Fig. 1. A Boltzmann viscoelastic plane containing multiple circular holes and elastic inclusions, subjected to time-dependent loading.

2. A direct boundary integral method in time domain

The Boltzmann model (see Fig. 1) contains two elastic components: instantaneous elasticity (elastic modulus: E_e) and viscous elasticity (elastic modulus: E_{ve}), and one viscous component (viscosity: η).

2.1. Basic equations

The system of integral equations for the problem is obtained by considering the superposition of two separate problems: (i) an infinite viscoelastic plane with circular holes and (ii) circular elastic discs.

For the first problem (the viscoelastic plane), we adopt the formula developed by Mesquita and Coda [2],

* Corresponding author. Tel.: +1 (612) 626 8768; Fax: +1 (612) 626 7750; E-mail: Huan0238@umn.edu

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$$u_i(\xi) + \gamma\dot{u}_i(\xi) = \frac{E_e + E_{ve}}{E_{ve}} \int_S U_{ij}(\xi, x) t_j(x) dS(x) - \int_S T_{ij}(\xi, x) u_j(x) dS(x) + \gamma\left[\int_S U_{ij}(\xi, x)\dot{t}_j(x) dS(x) - \int_S T_{ij}(\xi, x)\dot{u}_j(x) dS(x)\right] \quad (1)$$

where S is the totality of the boundaries of circular holes and $t_j(x)$, $u_j(x)$ and $\dot{t}_j(x)$, $\dot{u}_j(x)$ are the tractions, displacements and their rates. $U_{ij}(\xi,x)$ and $T_{ij}(\xi,x)$ are the Kelvin fundamental solutions and $\gamma = \eta/E_{ve}$.

For the second problem (the elastic discs), we use Somigliana's formula to express the displacements $u_i(\xi)$ at a point ξ within an elastic region Ω in terms of integrals of the tractions $t_j(x)$ and displacements $u_j(x)$ over the boundary S of the region,

$$u_i(\xi) = \int_S U_{ij}(\xi, x) t_j(x) dS(x) - \int_S T_{ij}(\xi, x) u_j(x) dS(x) \quad (2)$$

Boundary integral equations can be obtained from Eqs. (1) and (2) by taking the limit $\xi \to \xi_0 \in S$. We will perform the operation of taking this limit after the approximation of the unknown functions in Eqs. (1) and (2) and calculation of the integrals.

2.2. Time-stepping approach

To apply the time-stepping process, we assume linear behavior in time [2]:

$$\dot{u}_{i,s}(\xi) = \frac{u_{i,s}(\xi) - u_{i,s-1}(\xi)}{\Delta t}; \; \dot{t}_{i,s}(\xi) = \frac{t_{i,s}(\xi) - t_{i,s-1}(\xi)}{\Delta t} \quad (3)$$

where Δt is the time step size and $u_{i,s}(\xi)$, $t_{i,s}(\xi)$ and $u_{i,s-1}(\xi)$, $t_{i,s-1}(\xi)$ are the displacements and tractions in the i-direction at step s and $s-1$ respectively. We assume that the medium is initially free of stress and displacement, thus $u_{i,0}(\xi) = 0$, $t_{i,0}(\xi) = 0$.

After substituting Eq. (3) into Eq. (1), and performing some algebraic manipulation, we obtain the following expression:

$$u_{i,s}(\xi) = c_1 \int_S U_{ij}(\xi, x) t_{j,s}(x) dS(x) - \int_S T_{ij}(\xi, x) u_{j,s}(x) dS(x) + c_2\left[\int_S T_{ij}(\xi, x) u_{j,s-1}(x) dS(x) - \int_S U_{ij}(\xi, x) t_{j,s-1}(x) dS(x) + u_{i,s-1}(\xi)\right] \quad (4)$$

where we have introduced following abbreviations:

$$c_1 = \frac{\Delta t(E_e + E_{ve}) + \gamma E_{ve}}{(\Delta t + \gamma) E_{ve}}; \; c_2 = \frac{\gamma}{\Delta t + \gamma} \quad (5)$$

When computing results for step s, the displacements and tractions at step $s-1$ are presumed to be known.

2.3. Fourier series representation

For circular boundaries, the boundary displacements and tractions at each step s can be expanded in truncated Fourier series as follows:

$$u_{i,s}(R,\theta) = \frac{1}{2} a_{0,s}(u_i) + \sum_{n=1}^{N_s} \left[a_{n,s}(u_i)\cos n\theta + b_{n,s}(u_i)\sin n\theta\right]$$

$$t_{i,s}(R,\theta) = \frac{1}{2} a_{0,s}(t_i) + \sum_{n=1}^{N_s} \left[a_{n,s}(t_i)\cos n\theta + b_{n,s}(t_i)\sin n\theta\right] \quad (6)$$

The number of terms N_s in Eq. (6) may be different for each step s, and can be adjusted to obtain a specified computational accuracy [1].

Consider first the case of one hole. With the substitution of Eq. (6) into Eq. (4), and performing the analytical integration (by using complex variables and the residue theorem [3]), the displacements $u_{i,s}(\xi)$ can be expressed in terms of the Fourier coefficients for boundary tractions and displacements at step s and step $s-1$. Taking the limit $\xi \to \xi_0 \in S$, $u_{i,s}(\xi)$ can be represented by Fourier coefficients for boundary displacements directly, as in Eq. (6). This equivalence enables us to find the relationships between Fourier coefficients for tractions and displacements at the boundary of the circular hole.

For the case of multiple holes, the Fourier coefficients for displacements for one typical hole are written in terms of not only the Fourier coefficients for tractions at the boundary of the current hole, but also the coefficients for tractions and displacements at the boundaries of all other holes. The integrals involving the effects of other holes can be evaluated analytically [1].

The case of an elastic disc is treated similarly. As the result, the relationship between the Fourier coefficients for the tractions and displacements at the boundary of each disc can be found.

2.4. Coupling of the viscoelastic plane and elastic inclusions

At each step, a superposition process is used to handle the situation in which inclusions are introduced into the viscoelastic plane (a hole is treated as an inclusion with zero elastic modulus). For the viscoelastic matrix, we represent the total, or resultant, stresses $\sigma_{ij,s}^{(total)}$ as the sum of the initial stresses $\sigma_{ij,s}^{(0)}$ and the stress changes $\sigma_{ij,s}$ due to the presence of the holes, i.e.

$$\sigma_{ij,s}^{(total)} = \sigma_{ij,s}^{(0)} + \sigma_{ij,s} \quad (7)$$

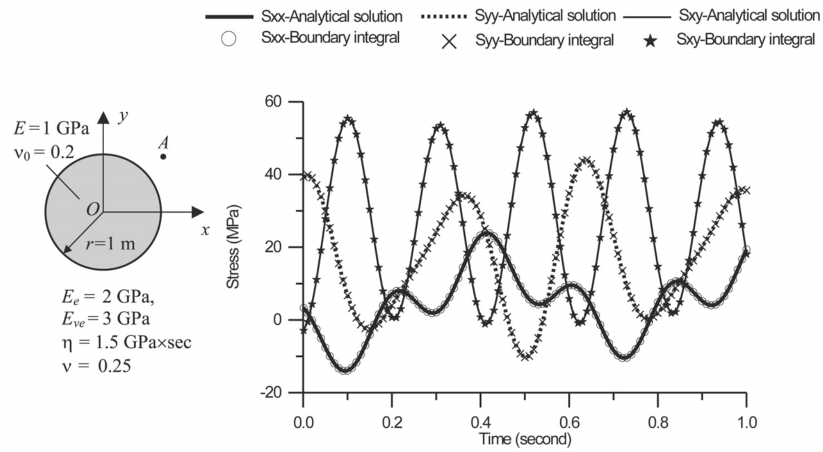

Fig. 2. Stresses at point A in the example of one inclusion.

where the components of $\sigma_{ij,s}$ at step s can be obtained by differentiating displacements given by Eq. (4) with respect to the co-ordinates ξ_x and ξ_y to compute the strains, and then substituting the strains into the constitutive relation of Boltzmann model. $\sigma_{ij,s}^{(0)}$ is the current stress state of an infinite, homogeneous viscoelastic plane, without any hole or inclusion. Thus, it has the same value as σ^{∞}, which is given in sinusoidal form in the present research:

$$\begin{aligned}\sigma_{xx}^{\infty}(t) &= \bar{\sigma}_{xx} + A_{xx}\sin(\omega_{xx}t - \phi_{xx})\\ \sigma_{yy}^{\infty}(t) &= \bar{\sigma}_{yy} + A_{yy}\sin(\omega_{yy}t - \phi_{yy})\\ \sigma_{xy}^{\infty}(t) &= \bar{\sigma}_{xy} + A_{xy}\sin(\omega_{xy}t - \phi_{xy})\end{aligned} \tag{8}$$

where, $\bar{\sigma}_{\mathrm{ij}}$, A_{ij}, ω_{ij} and ϕ_{ij} are the average, the varying amplitude, the frequency and the phase delay of stress σ_{ij}^{∞} respectively.

Similar relationships can be written for the displacement components of the matrix, namely:

$$u_{i,s}^{(total)} = u_{i,s}^{(0)} + u_{i,s} \tag{9}$$

where the components of $u_{i,s}$ are given by Eq. (4), and the components of $u_{i,s}^{(0)}$ are given by applying the same stress history σ^{∞} to an infinite homogeneous viscoelastic plane. Thus, $u_{i,s}^{(0)}$ can be obtained by applying the correspondence principle and Laplace transform to the solution for a corresponding problem of an infinite elastic plane subjected to constant stresses at infinity.

The continuity of tractions and displacements at the interface between matrix and inclusions provides the supplemental equations to define the unknown coefficients for the tractions and displacements at the boundaries of the holes and discs. The final equation system at each step is solved by an iteration procedure.

3. Examples

3.1. A benchmark problem of one inclusion

Consider the case of one circular elastic inclusion (Fig. 2) under the following time-dependent stresses at infinity:

$$\begin{aligned}\sigma_{xx}^{\infty}(t) &= 10 + 10\sin(10t + 10)\ \text{MPa}\\ \sigma_{yy}^{\infty}(t) &= 20 + 20\sin(20t + 20)\ \text{MPa}\\ \sigma_{xy}^{\infty}(t) &= 30 + 30\sin(30t + 30)\ \text{MPa}\end{aligned} \tag{10}$$

The analytical solution for this problem can be obtained by using the correspondence principle and Laplace transforms. We computed the stresses and displacements inside the matrix and inside the inclusion using our boundary integral method and compared the results with the analytical solutions. It can be seen from Fig. 2 that the numerical results agree well with the analytical solutions for all time steps.

3.2. An example with multiple holes and inclusions

This example involves the system of two circular holes and two circular inclusions inside the infinite viscoelastic plane (Fig. 3). The time-dependent biaxial stresses at infinity are given below:

$$\begin{aligned}\sigma_{xx}^{\infty}(t) &= 20 + 20\sin(2t + 20)\ \text{MPa}\\ \sigma_{yy}^{\infty}(t) &= 10 + 10\sin(t + 10)\ \text{MPa}\end{aligned} \tag{11}$$

The analytical solution for this problem is not available and commercial finite element software (ANSYS) was employed for the sake of comparison. We used the same time step length $\Delta t = 0.005$ s in both methods. With the direct boundary integral method, this problem

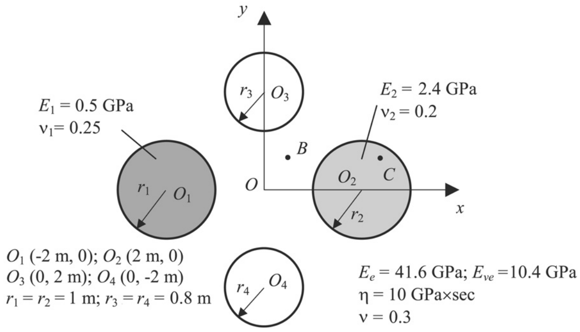

Fig. 3. Two holes and two inclusions.

required 11–12 terms of the Fourier series for the two inclusions and 10 terms for the two holes (given accuracy parameter $\delta = 10^{-5}$ [1]), and the computation took only 8 minutes with a Pentium III 931 MHz PC. The computation with ANSYS used 4176 elements and took about 5 hours with an IBM SP workstation.

It can be seen from Fig. 4 that the results (the displacement u_x at points B and C) are essentially identical.

4. Conclusions and future work

This paper presents a time-domain direct boundary integral method to solve the problem of an infinite viscoelastic plane containing multiple circular holes and elastic inclusions under time-dependent loading at infinity. The direct analytical evaluation of all the integrals involved ensures the high accuracy and efficiency of this method, as demonstrated by the examples. Applications of this approach lie in the area of composite materials. Future developments include (i) incorporation of a fast solver algorithm (e.g. the fast multipole method), to ease the computation involving a large number (e.g. thousands) of holes and inclusions; and (ii) incorporation of a finite exterior boundary for the viscoelastic plane to model practical problems for composite materials more accurately.

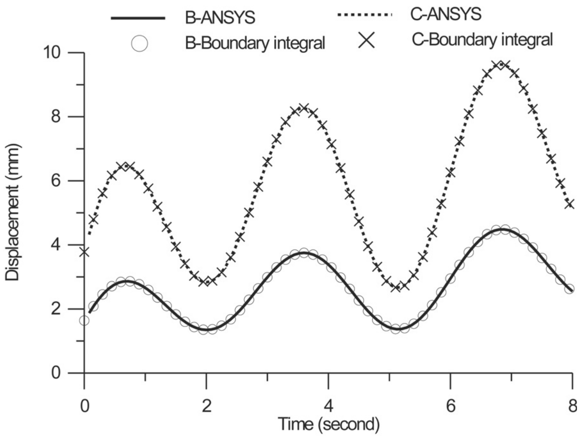

Fig. 4. u_x at points B and C for the example of two holes and two inclusions.

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References

[1] Crouch SL, Mogilevskaya SG. On the use of Somigliana's formula and Fourier series for elasticity problems with circular boundaries. Int J for Num Methods Eng 2003;58:537–578.

[2] Mesquita AD, Coda HB. Boundary integral equation method for general viscoelastic analysis. Int J Solids Struct 2002;39:2643–2664.

[3] Mogilevskaya SG, Crouch SL. A Galerkin boundary integral method for multiple circular elastic inclusions. Int J Num Methods Eng 2001;52:1069–1106.