A posteriori error estimates for an eigenvalue problem arising from fluid–structure interaction

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Abstract

In this paper, we introduce a reliable and efficient a posteriori error estimator for the approximation of an eigenvalue problem using Brezzi–Douglas–Marini finite element spaces of any order. According to the author's knowledge, it is the first time that an a posteriori error analysis for mixed approximation of an eigenvalue problem is developed. Indeed, other authors see Duran et al. in [1] used the equivalence between the mixed finite element method of Raviart-Thomas of the lowest order and the non-conforming piece-wise linear approximation of Crouzeix and Raviart.

Keywords: A posteriori error estimator; Eigenvalue problem; Fluid-structure interaction; Mixed finite element

1. Introduction

In this paper, we present an a posteriori error estimator for the Brezzi–Douglas–Marini approximation of an eigenvalue problem that arises from the displacement formulation to compute the vibration modes of an acoustic fluid contained within a rigid cavity. We define an error estimator of the residual type and prove that, under some conditions on the regularity of the continuous eigensolution, it is equivalent to the H (div)-norm of the error up to higher-order terms. The constants involved in this equivalence depend on the corresponding eigenvalue but are independent of the mesh size. Moreover, the square root of the error in the approximation of the eigenvalues is also bounded by a constant multiplied by the estimator. These results are exposed in a concise way in Theorem (11).

2. Statement of the problem

We consider the following eigenvalue problem: find $\lambda \in \mathbb{C}$ such that there exists $\boldsymbol{u} \neq \mathbf{0}$ such that

$$\begin{cases} -\nabla \operatorname{div} \boldsymbol{u} = \lambda \boldsymbol{u} & \text{in } \Omega \\ \operatorname{rot} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

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Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

where $\Omega \subset \mathbb{R}^2$ is a simply connected polygonal domain, $\partial\Omega$ is its boundary and n is its outward normal unit vector.

This problem has been studied by many authors concerning fluid–structure interaction [2, 3]. Moreover, since in two dimensions the divergence and rotational operators are isomorphic, it is equivalent to Maxwell's eigenproblem for a cavity resonator with dielectric constant ε and magnetic permeability μ constant and equal to 1 [4,5].

We shall use the standard notation for the Sobolev spaces $H^m(\Omega)$, their norms $\| \, \|_m$ and seminorms $| \, |_m$.

A variational formulation of Eq. (1) reads: find $\lambda \in \mathbb{C}$ s.t. there exists $\boldsymbol{u} \in H_0(\operatorname{div}, \Omega)$, with $\mathbf{u} \neq \mathbf{0}$:

$$\begin{cases} (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) = \lambda(\boldsymbol{u}, \boldsymbol{v}) & \forall \boldsymbol{v} \in H_0(\operatorname{div}, \Omega) \\ (u, \mathbf{rot} q) = 0 & \forall q \in H_0^1(\Omega), \end{cases} \tag{2}$$

where (,) denotes the L^2-inner product and $H_0(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^2 \colon \operatorname{div} \boldsymbol{v} \in L^2(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$ is endowed with the norm $\|\mathrm{v}\|_{\operatorname{div}}^2 = \|\mathrm{v}\|_0^2 + \|\operatorname{div} \mathrm{v}\|_0^2$. It is well known that Eq. (2) admits a countable set of real and positive eigenvalues. Moreover, the eigenfunctions satisfy $\boldsymbol{u} \in H^s(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in [H^s(\Omega)]^2 : \operatorname{div} \boldsymbol{v} \in H^s(\Omega)\}$, for some $s > \frac{1}{2}$ depending on Ω ($s = 1$ when Ω is convex) [6].

Let $\{\mathcal{J}_h\}$ be a regular family of triangulations of Ω, where as usual h denotes the maximum diameter of the elements K in $\mathcal{J}_h$. The Brezzi–Douglas–Marini spaces are defined for $k \geq 1$ by

$$BDM_k = \{v \in H\,(\text{div}, \Omega) : v|_K \in [P_k(K)]^2 \quad \forall K \in \mathcal{J}_h\},$$

where $P_k(K)$ denotes the space of polynomial of degree at most k on K [7].

Setting $V_h = BDM_k \cap H_0(\text{div}, \Omega)$, and denoting by Q_h the subspace of $H_0^1(\Omega)$ consisting of continuous piecewise polynomial of degree at most $k + 1$, the discrete problem is then given by: find $\lambda_h \in \mathbb{R}$ s.t. there exists $\boldsymbol{u}_h \in V_h$, with $\boldsymbol{u}_h \neq \mathbf{0}$:

$$\begin{cases} (\text{div}\,\boldsymbol{u}_h,\ \text{div}\,\boldsymbol{v}) = \lambda_h(\boldsymbol{u}_h,\ \boldsymbol{v}) & \forall \boldsymbol{v} \in V_h \\ (\boldsymbol{u}_h,\ \mathbf{rot}\,q) = 0 & \forall q \in Q_h \end{cases} \tag{3}$$

Let $(\lambda, \boldsymbol{u})$ be an eigensolution of Eq. (2), such that λ is a simple eigenvalue and $\|\boldsymbol{u}\|_0 = 1$. It follows from the abstract theory [8, 9] and known a priori estimates that, for h small enough (depending on λ), there exists $(\lambda_h, \boldsymbol{u}_h)$ eigenpair of Eq. (3) with $\|\boldsymbol{u}_h\|_0 = 1$, such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\text{div}} = O(h^t) \tag{4}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 = O(h^r) \tag{5}$$

$$|\lambda - \lambda_h| = O(h^{2t}) \tag{6}$$

where $t = \min\{s, k\}$ and $r = \min\{s, k + 1\}$.

Our goal is to estimate a posteriori the error in the approximation of the eigensolutions. From now on, we will denote by $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$ the error in the approximation of the eigenfunctions.

3. Residual-based a posteriori error estimator

In this section, we present the error estimator and state the results that allow us to prove that it is equivalent, up to higher-order terms, to the H (div)-norm of the error, when the continuous eigensolution $\boldsymbol{u}$ is smooth enough. We will report the details of the proofs in a forthcoming paper.

First, we introduce some notation. Let $\mathcal{E}$ be the set of the interior edges of the mesh and $\mathcal{E}_K \subset \mathcal{E}$ be the subset of edges of K.

The following lemmas provide the residual equations that will be the starting points of our error analysis:

Lemma 1 *For* $\boldsymbol{v} \in H_0$ (div, Ω) *there holds*

$$(\text{div}\,\boldsymbol{e}_h,\ \text{div}\,\boldsymbol{v}) - (\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h,\ \boldsymbol{v}) = -(\text{div}\,\boldsymbol{u}_h,\ \text{div}\,\boldsymbol{v}) + \lambda_h(\boldsymbol{u}_h,\ v)$$

$$= \sum_{k\in\mathcal{J}_h}\left[(\boldsymbol{r}_1,\ \boldsymbol{v})_K - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e [\![\text{div}\,\boldsymbol{u}_h]\!]\boldsymbol{v}\cdot\boldsymbol{n})\right], \tag{7}$$

where $\boldsymbol{r}_1 = \nabla\,\text{div}\,\boldsymbol{u}_h + \lambda_h\boldsymbol{u}_h$ *is the residual of the first equation of Eq. (1).*

Lemma 2 *For* $q \in H_0^1$ (Ω) *there holds*

$$(e_h,\ \mathbf{rot}\,q) = \sum_{K\in\mathcal{J}_h}\left[(r_2,\ q)_K + \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e q[\![\boldsymbol{u}_h\cdot\boldsymbol{t}]\!]\right], \tag{8}$$

where $r_2 = -\,\text{rot}\,\boldsymbol{u}_h$ *is the residual of the second equation of Eq. (1) and, for each triangle* K, $\mathbf{t}$ *denotes its unit tangent vector oriented counterclockwise.*

For any $K \in \mathcal{J}_h$, we define two local error indicators by

$$\eta_{1,K}^2 = h_K^2\|\boldsymbol{r}_1\|_{0,K}^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_K} h_e\|[\![\text{div}\,u_h]\!]\|_{0,e}^2,$$

$$\eta_{2,K}^2 = h_K^2\|r_2\|_{0,K}^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_K} h_e\|[\![\boldsymbol{u}_h\cdot\boldsymbol{t}]\!]\|_{0,e}^2,$$

and the corresponding error estimators by

$$\eta_1 = \left(\sum_{K\in\mathcal{J}_h}\eta_{1,K}^2\right)^{\frac{1}{2}},$$

$$\eta_2 = \left(\sum_{K\in\mathcal{J}_h}\eta_{2,K}^2\right)^{\frac{1}{2}}.$$

Then the following propositions hold true:

Proposition 3 *There exists a positive constant* C, *independent of* h, *such that*

$$\|\boldsymbol{e}_h\|_0 \le C\|\text{div}\,\boldsymbol{e}_h\|_0 + C\eta_2. \tag{9}$$

Proposition 4 *There holds*

$$\|\text{div}\,\boldsymbol{e}_h\|_0^2 \le C_\lambda\|\text{div}\,\boldsymbol{e}_h\|_0(\eta_1 + \eta_2) + C_\lambda\eta_2^2 + (\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h, \boldsymbol{e}_h), \tag{10}$$

where C_λ *is a positive constant dependent on* λ.

Since $\|\boldsymbol{u}\|_0 = \|\boldsymbol{u}_h\|_0 = 1$, the last term in Eq. (10) can be written as

$$(\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h,\ \boldsymbol{e}_h) = (\lambda + \lambda_h)[1 - (\boldsymbol{u},\ \boldsymbol{u}_h)] = \frac{\lambda + \lambda_h}{2}\|\boldsymbol{e}_h\|_0^2 \tag{11}$$

and hence if the continuous eigensolution is smooth enough (i.e. $\boldsymbol{u} \in H^s(\Omega, \text{div})$ for some $s \ge k + 1$), then by the a priori estimate in Eq. (5), it turns out to be higher order than $\|\,\text{div}\,\boldsymbol{e}_h\|_0^2$. In order to have that η_2^2 is a higher-order term as well, it is enough to prove that the local error indicator $\eta_{2,K}$, is efficient, namely such that $\eta_{2,K} \le C\,\|\boldsymbol{e}_h\|_{0,K^*}$, where K^* denotes the union of all elements sharing an edge with K and C is a constant depending only on the regularity of the elements of K^*. Indeed, if $\eta_{2,K} \le C\,\|\boldsymbol{e}_h\|_{0,K^*}$, then $\eta_2 \le C\,\|\boldsymbol{e}_h\|_0$, with C constant depending only on the regularity of the mesh.

3.1. Local upper bound for $\eta_{2,\mathrm{K}}$

In this section, we show that the error indicator $\eta_{2,K}$, is bounded above by the L^2-norm of the error in the neighbourhood of the element K. Indeed, the following results hold true:

Proposition 5 *There exists a constant* C, *depending only on the regularity of the element* K, *such that*

$$h_K^2\|r_2\|_{0,K}^2 \leq C\|\boldsymbol{e}_h\|_{0,K}^2.$$

For any interior edge $\bar{e} \in \mathcal{E}$, let $K_{\bar{e}}^1$ and $K_{\bar{e}}^2$ denote the two elements in $\mathcal{J}_h$ sharing $\bar{e}$.

Proposition 6 *Let* $\bar{\mathrm{e}} \in \mathcal{E}$. *There exists a constant* C, *depending only on the regularity of* $\mathrm{K}_{\bar{\mathrm{e}}}^1$ *and* $\mathrm{K}_{\bar{\mathrm{e}}}^2$, *such that*

$$\frac{1}{2}h_{\bar{e}}\|[\![\boldsymbol{u}_h \cdot \boldsymbol{t}]\!]\|_{0,\bar{e}}^2 \leq C\|\boldsymbol{e}_h\|_{0,K_{\bar{e}}^1\cup K_{\bar{e}}^2}^2.$$

We can therefore state the following result:

Theorem 7 *There exists a constant* C, *depending only on the regularity of the mesh, such that*

$$\eta_2 \leq C\|\boldsymbol{e}_h\|_0.$$

Moreover, the following local estimate holds:

$$\eta_{2,K} \leq C\|\boldsymbol{e}_h\|_{0,K^*},$$

with C *constant depending only on the regularity of the elements* K *of* K*.

3.2. Local upper bound for $\eta_{1,\mathrm{K}}$

In this section, we show that the error indicator $\eta_{1,K}$ is bounded above by the L^2-norm of the divergence of the error in the neighbourhood of the element K. This, together with the result of the previous section, yields the efficiency of the error indicator $\eta_{1,K} + \eta_{2,K}$. Indeed, the following results hold:

Proposition 8 *There exists a constant* C, *depending only on the regularity of* K, *such that*

$$h_K\|\boldsymbol{r}_1\|_{0,K} \leq C\Big(\|\mathrm{div}\ \boldsymbol{e}_h\|_{0,K}+h_K\|\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h\|_{0,K}\Big).$$

Proposition 9 *Let* $\bar{\mathrm{e}} \in \mathcal{E}$. *There exists a constant* C, *depending only on the regularity of* $\mathrm{K}_{\bar{\mathrm{e}}}^1$ *and* $\mathrm{K}_{\bar{\mathrm{e}}}^2$, *such that*

$$\frac{1}{2}h_{\bar{e}}^{\frac{1}{2}}\|[\![\mathrm{div}\ \boldsymbol{u}_h]\!]\|_{0,\bar{e}} \leq C\Big(\|\mathrm{div}\ \boldsymbol{e}_h\|_{0,K_{\bar{e}}^1\cup K_{\bar{e}}^2}+h_{\bar{e}}\|\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h\|_{0,K_{\bar{e}}^1\cup K_{\bar{e}}^2}\Big).$$

As a consequence of the above propositions, we state the following theorem:

Theorem 10 *There exists a constant* C, *depending only on the regularity of the elements of* K*, *such that*

$$\eta_{1,K} \leq C\Big(\|\mathrm{div}\ \boldsymbol{e}_h\|_{0,K^*}+h_K\|\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h\|_{0,K^*}\Big).$$

The term $h_K\|\lambda\boldsymbol{u} - \lambda_h\boldsymbol{e}_h\|_{0,K^*}$ in the previous theorem is a higher-order term. Indeed, for each element $K' \in \mathcal{J}_h$

$$\begin{aligned} h_{K'}\|\lambda\boldsymbol{u} - \lambda_h\boldsymbol{u}_h\|_{0,K'} &\leq |\lambda - \lambda_h|h_{K'}\|\boldsymbol{u}_h\|_{0,K'}+\lambda h_{K'}\|\boldsymbol{e}_h\|_{0,K'} \\ &\leq Ch^{2t+1} + \lambda h_{K'}\|\boldsymbol{e}_h\|_{0,K'}, \end{aligned}$$

the last inequality because of the a priori estimate in Eq. (6). Note that the right-hand side is asymptotically negligible with respect to the local error $\|\mathrm{div}\ \boldsymbol{e}_h\|_{0,K'}$.

Putting together the results of theorems 7 and 10, we show that the error indicator $\eta_{1,K} + \eta_{2,K}$ is bounded above by the local error, up to a multiplicative constant and higher-order terms, namely,

$$\eta_{1,K} + \eta_{2,K} \leq C\|\boldsymbol{e}_h\|_{\mathrm{div},K^*}+O(h^{2t+1}) + O(h_K)\|\boldsymbol{e}_h\|_{0,K^*}.$$

We are now ready to state the main result of our error analysis.

Theorem 11 *Let us assume that* $\mathbf{u} \in \mathrm{H}^{\mathrm{k}+1}$ (div, Ω). *Then there exist two constants* C_λ^1 *and* C_λ^2, *depending on* λ *and on the regularity of the mesh, and a constant* C *depending only on the regularity of the mesh, such that*

$$\|\boldsymbol{e}_h\|_{\mathrm{div}} \leq C_\lambda^1(\eta_1 + \eta_2) + h.o.t. \quad (12)$$

$$\eta_{1,K} + \eta_{2,K} \leq C\|\boldsymbol{e}_h\|_{\mathrm{div},K^*}+h.o.t. \quad (13)$$

$$|\lambda - \lambda_h|^{\frac{1}{2}} \leq C_\lambda^2(\eta_1 + \eta_2) + h.o.t. \quad (14)$$

Therefore, it follows from Eqs (12) and (14) that our estimator is reliable. On the other hand, Eq. (13) tell us that $\eta_{1,K} + \eta_{2,K}$ is an efficient local error indicator, in the sense that when $\eta_{1,K} + \eta_{2,K}$ is large, then the error in the vicinity of the element K must also be large. Hence, $\eta_{1,K} + \eta_{2,K}$ can be used as the basis of an adaptive refinement algorithm.

References

[1] Durán G, Gastaldi L, Padra C. A posteriori error estimators for mixed approximations of eigenvalue problems. Math Models Methods Appl Sci 1999;9:1165–1178.

[2] Bathe KJ, Wang X. On mixed elements for acoustic fluid-structure interaction. M^3 AS 1997;7:329–344.

[3] Bermúdez A, Durán R, Muschietti A, Rodríguez R, Solomin J. Finite element vibration analysis of fluid-solid systems without spurious modes. SIAM J Numer Anal 1995;32:1280–1295.

[4] Boffi D, Fernandes P, Gastaldi L, Perugia I. Computational models of electromagnetic resonators: analysis of edge element approximation. SIAM J Numer Anal 1998;36:1264–1290.

[5] Hiptmair R. Finite elements in computational electromagnetism. Acta Numerica 2002;11:237–339.

[6] Amrouche C, Bernardi C, Dauge M, Girault V. Vector potential in three-dimensional nonsmooth domains. Math Methods Appl Sci 1998;21:823–864.

[7] Brezzi F, Fortin M. Mixed and Hybrid Finite Element Methods. New York: Springer-Verlag, 1991.
[8] Boffi D, Brezzi F, Gastaldi L. On the convergence of eigenvalues for mixed formulations. Ann Sc Norm Super Pisa Cl Sci 1997;25:131–154.
[9] Babuška I, Osborn J. Eigenvalue problems. In: Handbook of Numerical Analysis, Vol II (PG Ciarlet, JL Lions, editors, Amsterdam: North-Holland, 1991, pp. 641–787.