Dynamic analysis of stochastic structures using the random factor method

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Abstract

A new method called the random factor method (RFM) for the dynamic analysis of stochastic truss structures is presented in this paper. Using the RFM, the structural physical parameters and geometry can be considered as random variables. The structural stiffness and mass matrices can then be respectively divided into the product of two parts corresponding to the random factors and the deterministic matrix. Computational expressions for the numerical characteristics of the natural frequencies and mode shapes are derived using the algebra synthesis method. The influences of the randomness of the structural parameters on the dynamic characteristics are demonstrated using a truss structure.

Keywords: Random factor method; Stochastic truss structures; Natural frequency; Mode shape; Random variable; Numerical characteristics

1. Introduction

It is very important to compute the dynamic characteristics of structures with uncertainty in their structural parameters arising from manufacturing tolerances, material defects and variation in operating conditions. In many cases, the Monte-Carlo simulation method [1] and the perturbation method [2] have often been used for structural dynamic analysis. However, the Monte-Carlo method needs a large amount of computation. The perturbation method can not reflect the effect of the individual parameters on the structural dynamic response.

In this paper, the dynamic characteristics analysis of stochastic structures is investigated, and a new method called the random factor method is proposed. Truss structures are used to illustrate examples of this method, in which the randomness of the structural physical parameters (Young's modulus and mass density) and geometry (length and cross-area of bar) are considered.

The procedure for the random factor method (RFM) is as follows. Firstly, a structural parameter variable with uncertainty is expressed as a random factor multiplied by the mean value of this structural parameter. Secondly, the structural mass and stiffness matrices are expressed as random factors of the structural parameters multiplied by their mean value respectively. Finally, the dynamic characteristics are expressed as the functions of these random factors. Therefore, the effect of the randomness of the structural parameters on the natural frequencies and mode shapes can be easily identified, and compared to other methods, the computational work to obtain the dynamic characteristics is very small.

2. Dynamic characteristics analysis using the random factor method

Suppose that there are n elements in the truss structure under consideration. The mass matrix $[M]$ and stiffness matrix $[K]$ of truss structure in global coordinates can be respectively expressed as:

$$[M] = \sum_{e=1}^{n} [M_e] = \sum_{e=1}^{n} \frac{1}{2} \rho_e A_e l_e [I] \tag{1}$$

$$[K] = \sum_{e=1}^{n} [K_e] = \sum_{e=1}^{n} \frac{E_e A_e}{l_e} [G] \tag{2}$$

where $[K_e]$ and $[M_e]$ are the stiffness and mass matrices, and E_e, A_e, l_e and ρ_e are the Young's modulus, cross-

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sectional area, length and mass density respectively of the eth element. $[I]$ is a sixth-order identity matrix, and $[G]$ is a 6×6 matrix.

The structural physical parameters (ρ_e, E_e) and the geometric dimensions (A_e, l_e) are simultaneously considered as random variables. However, the randomness of the eth element is considered to be the same for all elements. Hence, the Young's modulus, mass density, length and cross-sectional area can be written respectively as: $E_e = \tilde{E}\bar{E}$, $\rho_e = \tilde{\rho}\bar{\rho}$, $l_e = \tilde{l}\bar{l}_e$ and $A_e = \tilde{A}\bar{A}_e$ ($e = 1, \ldots, n$). $\bar{E}$ and $\bar{\rho}$ are the mean values of the Young's modulus and density respectively. $\bar{l}_e$ and $\bar{A}_e$ are the mean values that denote the nominal length and cross-sectional area of the eth bar respectively. $\tilde{E}$, $\tilde{\rho}$, $\tilde{l}$ and $\tilde{A}$ are the random variable factors of the Young's modulus, density, length and cross-sectional area respectively. The random variable factors are each given a mean value (μ) and a mean variance (σ), for example, $\tilde{E} = \mu_{\tilde{E}} \pm \sigma_{\tilde{E}}$. In this analysis, the mean values of the random variable factors are taken to be 1.0, that is, $\mu_{\tilde{E}} = \mu_{\tilde{\rho}} = \mu_{\tilde{l}} = \mu_{\tilde{A}} = 1.0$.

From Eq. (1), it is easily seen that when ρ_e, A_e and l_e are random variables, $[M_e]$ and $[M]$ are also random variables. Let:

$$[M_e] = \tilde{\rho}\tilde{A}\tilde{l}[\bar{M}_e] \tag{3}$$

where $[\bar{M}_e]$ is the deterministic part of the mass matrix $[M_e]$. The expression shows that the mass matrix $[M_e]$ can be divided into the product of two parts, corresponding to the random variables $\tilde{\rho}$, $\tilde{A}$, $\tilde{l}$, and the constant matrix $[\bar{M}_e]$. The randomness of $[M_e]$ is only dependent on the random variable factors $\tilde{\rho}$, $\tilde{A}$ and $\tilde{l}$. Constructing the deterministic matrix $[\bar{M}_e]$ is same as constructing the mass matrix in Eq. (1) for the eth element, and taking the parameters as $l_e = \bar{l}_e$, $A_e = \bar{A}_e$, $\rho_e = \bar{\rho}$. Hence, $[M]$ in Eq. (1) can now be written as:

$$[M] = \sum_{e=1}^{n}[M_e] = \sum_{e=1}^{n}(\tilde{\rho}\tilde{A}\tilde{l}[\bar{M}_e]) = \tilde{\rho}\tilde{A}\tilde{l}[\bar{M}] \tag{4}$$

Likewise, from Eq. (2), $[K]$ is a random variable and can be written as:

$$[K] = \sum_{e=1}^{n}[K_e] = \sum_{e=1}^{n}(\frac{\tilde{E}\tilde{A}}{\tilde{l}}[\bar{K}_e]) = \frac{\tilde{E}\tilde{A}}{\tilde{l}}[\bar{K}] \tag{5}$$

where $[\bar{K}_e]$ and $[\bar{K}]$ are the deterministic part (mean values) of the stiffness matrices $[K_e]$ and $[K]$ respectively.

In the perturbation stochastic finite element method (PSFEM) and perturbation method, the stiffness and mass matrices can also be expressed as, $[M] = [\bar{M}] + \varepsilon_1 [\bar{M}]$, $[K] = [\bar{K}] + \varepsilon_2 [\bar{K}]$. However, these small parameters ε_1 and ε_2 are not random variables, so they cannot directly reflect the randomness of the individual structural parameters. In the random factor method (RFM), the structural parameter variable and its random factor obey the same probabilistic distribution. Hence, unlike the PSFEM and perturbation method, using the random factor method, the randomness of the individual parameters (E, A, l and ρ) can be examined.

By using the random factor method, the jth natural frequency ω_j and mode shape $\{\phi\}_j$ can be respectively written as:

$$\omega_j = \tilde{\omega}_j\bar{\omega}_j \qquad \{\phi\}_j = \tilde{\phi}_j\{\bar{\phi}_j\} \tag{6}$$

By using the Rayleigh quotient expression, the jth natural frequency can be obtained:

$$\omega_j^2 = \frac{\{\phi\}_j^T[K]\{\phi\}_j}{\{\phi\}_j^T[M]\{\phi\}_j} = \frac{\tilde{\phi}_j\tilde{E}\tilde{A}\tilde{\phi}_j}{\tilde{\phi}_j\tilde{\rho}\tilde{A}\tilde{l}^2\tilde{\phi}_j}\frac{\{\bar{\phi}\}_j^T[\bar{K}]\{\bar{\phi}\}_j}{\{\bar{\phi}\}_j^T[\bar{M}]\{\bar{\phi}\}_j} =$$

$$\frac{\tilde{E}}{\tilde{\rho}\tilde{l}^2}\frac{\bar{K}_j}{\bar{M}_j} = \frac{\tilde{E}}{\tilde{\rho}\tilde{l}^2}\bar{\omega}_j^2;\ \omega_j = \sqrt{\frac{\tilde{E}}{\tilde{\rho}}}\frac{\bar{\omega}_j}{\tilde{l}};\ \tilde{\omega}_j = \sqrt{\frac{\tilde{E}}{\tilde{\rho}}}\frac{1}{\tilde{l}} \tag{7}$$

where $\bar{K}_j$, $\bar{M}_j$, $\bar{\omega}_j$ are all deterministic quantities corresponding to the jth-order stiffness, mass and natural frequency of the structure when the parameters are $A_e = \bar{A}_e$, $l_e = \bar{l}_e$, $E_e = \bar{E}$ and $\rho_e = \bar{\rho}$. From the deterministic stiffness and mass matrices $[\bar{K}]$ and $[\bar{M}]$, the deterministic values (mean values) of every order natural frequency $\bar{\omega}_j$ can be obtained by means of the conventional dynamic analysis method.

The numerical characteristics of the natural frequencies can be obtained by the algebra synthesis method [3,4]. It can be seen from Eq. (7) that the natural frequencies are not dependent on variation of the cross sectional area ($\sigma_{\tilde{A}}$), and are only dependent on the mean variance of the other parameters corresponding to $\sigma_{\tilde{E}}$, $\sigma_{\tilde{\rho}}$ and $\sigma_{\tilde{l}}$.

From the modal analysis theory, the modal matrix $[\phi]$ has the orthogonal property as follows:

$$[\phi]^T[M][\phi] = \tilde{\phi}^2\tilde{\rho}\tilde{A}\tilde{l}[\bar{\phi}]^T[\bar{M}][\bar{\phi}] = [I];$$

$$[\phi]^T[K][\phi] = \tilde{\phi}^2\frac{\tilde{E}\tilde{A}}{\tilde{l}}[\bar{\phi}]^T[\bar{K}][\bar{\phi}] = [\Omega] = \frac{\tilde{E}}{\tilde{\rho}\tilde{l}^2}diag[\bar{\omega}^2] \tag{8}$$

From Eq. (8), we can obtain the same result for the random component of each mode shape:

$$\tilde{\phi}_j = \tilde{\phi} = \frac{1}{\sqrt{\tilde{\rho}\tilde{A}\tilde{l}}} \tag{9}$$

It can be seen from Eq. (9) that the mode shapes are not dependent on randomness of the Young's modulus ($\sigma_{\tilde{E}}$), and are only dependent on the mean variance of

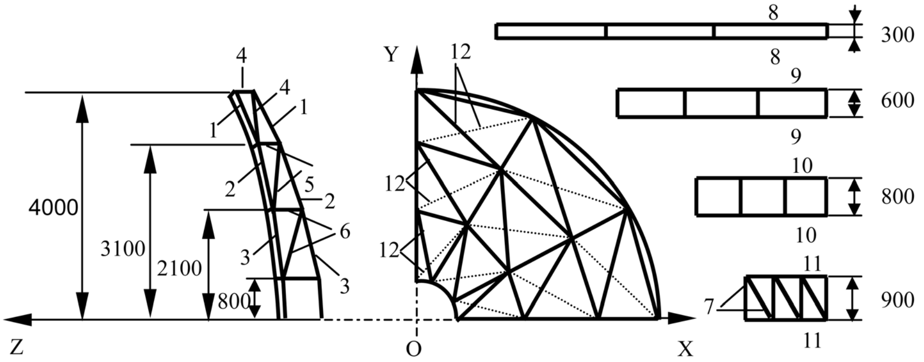

Fig. 1. Quarter of 8-meter caliber antenna (unit: mm).

the other parameters corresponding to $\sigma_{\tilde{\rho}}$, $\sigma_{\tilde{A}}$ and $\sigma_{\tilde{l}}$. The numerical characteristics of the mode shapes can be obtained by the algebra synthesis method [3,4].

3. Examples

The conventional dynamic characteristics of the deterministic truss structures can be analyzed by any FEM analysis software. In the following simulation, an 8-meter antenna structure shown in Fig. 1 is used as an example, with the material properties of steel. The mean values of the Young's modulus and density are respectively and $\bar{E} = 2.058 \times 10^5$(MPa) and $\bar{\rho} = 7.65 \times 10^3$(kg/m^3). The antenna is a 96-node and 336-element space truss structure, with 12 elements. The mean values of the cross-sectional area of each element are given in Table 1.

Table 1
The mean value of the cross-sectional area of each element

Elements	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12
Areas ($\times 10^{-4}$ m^2)	3.0	4.0	6.0	2.0	3.0	3.0	6.0	2.0	3.0	4.0	6.0	2.0

In the results, values from both the deterministic and random models are presented. In the deterministic model, the mean values of the random variables are unity, and their mean variance is zero. In the random model, in order to investigate the effect of the random variables E, ρ, l, and A on the structural dynamic characteristics, different combinations are presented. The computational results (mean value and mean variance (standard deviation)) for the first natural frequency and mode shape are given in Tables 2 and 3 respectively. In addition, in order to verify the effectiveness of the random factor method, the computational results that obtained by Monte-Carlo simulation method are also given in Tables 2 and 3, in which 3000 simulations are used.

Table 2
The computational results for the natural frequency

Mean variance	μ_{ω_1} (Hz)	σ_{ω_1} (Hz)
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$ (deterministic model)	22.818	0
$\sigma_{\tilde{A}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = 0$	22.818	0
$\sigma_{\tilde{E}} = 0.1$ $\sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$	22.818	1.1417
$\sigma_{\tilde{\rho}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$	22.818	1.1360
$\sigma_{\tilde{l}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{A}} = 0$	22.818	2.2275
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.1$	22.818	2.5233
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.2$	22.818	4.7588
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.2$*	22.904*	4.6992*

* Monte-Carlo simulation method

Table 3
The computational results for the mode shape

Mean variance	$\mu_{\phi_{11}}$ ($\times 10^{-3}$)	$\sigma_{\phi_{11}}$ ($\times 10^{-3}$)
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$ (deterministic model)	2.1190	0
$\sigma_{\tilde{E}} = 0.1$ $\sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$	2.1190	0
$\sigma_{\tilde{\rho}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0$	2.1190	0.1835
$\sigma_{\tilde{A}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = 0$	2.1190	0.1835
$\sigma_{\tilde{l}} = 0.1$ $\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{A}} = 0$	2.1190	0.1835
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.1$	2.1190	0.4493
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.01$	2.1190	0.0449
$\sigma_{\tilde{E}} = \sigma_{\tilde{\rho}} = \sigma_{\tilde{l}} = \sigma_{\tilde{A}} = 0.01$*	2.1219*	0.0460*

* Monte-Carlo simulation method

Comparing the first two rows in Table 2, it can be seen that randomness of the cross-sectional area does not have any effect on the mean natural frequency, as expected from Eq. (7). In rows 2 to 4 in Table 3, it is shown that when the mean variation of the parameters ρ, l and A are the same, the mean variance of mode shape value is also unchanged, as expected from Eq. (9). In both Tables 2 and 3, comparisons of the results obtained from the RFM with the Monte-Carlo method are very similar, by which the validity of the RFM is verified.

4. Conclusions

In this paper, the effect of uncertainty in the material parameters and structural dimensions on the randomness of the dynamic characteristics is presented using a new technique called the random factor method. The results from this method are in very good agreement with results obtained from the Monte-Carlo simulation method. In this work, the randomness of the eth element of a truss structure was considered to be the same for all elements. Future work will investigate randomness of the individual structural elements, and the effect on the natural frequencies and mode shapes.

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