The interval finite element method for static structural analysis

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Abstract

This paper focuses on possible solution strategies for the interval finite element method (IFEM) for mechanical structures with uncertain parameters. Particularly the static IFEM analysis is discussed, which was inspired from the fuzzy finite element method (FEM) for dynamic analysis developed by Moens [1]. For an implementation of the IFEM for statics, different solution strategies based on interval arithmetics, optimisation and vertex analysis were developed and tested.

Keywords: Uncertainty; Interval finite element analysis; Structural static analysis

1. Introduction

A reliable finite element (FE) analysis in a virtual prototyping environment can reduce the need for expensive physical prototypes. This assumes accurate FE models, which can incorporate non-deterministic information, such as properties subject to tolerances, environmental effects, non-uniform material properties, etc. Various probabilistic procedures have been developed for this purpose [2]. Since the statistical information on different design parameters is often limited at an early product design stage, the use of probabilistic FE approaches is not always appropriate [3]. The interval finite element method (IFEM) can be complementary to the stochastic FE methods when the input parameters can be bounded, but the likelihood (probability density function (PDF)) is unknown. In these cases, the IFEM can be useful in a worst-case oriented design optimisation. Based on the concept of fuzzy numbers [4], this method can be extended further to a fuzzy finite element method (FFEM). The most common procedure for the Fuzzy FE analysis is the α-cut strategy (Fig. 1). In this strategy, an internal finite element (IFE) solution is performed on different membership levels. Hence, the static IFE procedures presented here are readily extendable to the fuzzy approach. In a possibilistic interpretation, the fuzzy FEM can be useful in a reliability framework. By using the input membership functions to express a possibility distribution on uncertain model parameters, the fuzzy outcome can be used to predict a possibility of failure. Furthermore, the method can be a valuable robust design optimisation and tolerance analysis tool [5]. The application that is presented in this study will use a Monte Carlo (MC) analysis as a reference result, although the objectives and conditions of use of MC and IFE are quite different.

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2. IFEM for static analysis

Suppose we have a structural problem with imprecisely defined design parameters (denoted as $\{x\}$). In a displacement-based FE analysis, the output field is formed by the nodal displacements. In IFEM for statics, we are looking for a conservative hypercubic approximation of the solution set of the linear equation system:

$$\langle\{\boldsymbol{u}\}\rangle = \left\{\{u\}|\{u\} = \left[K(x)^{-1}F, \{x\} \in \{\boldsymbol{x}\}\right\}\right. \quad (1)$$

where $\{\boldsymbol{x}\}$ represents the interval input uncertainties, $\{\boldsymbol{u}\}$ the resulting interval displacement vector, $[K]$ the stiffness matrix and F the force vector. The stiffness matrix $[K]$ can depend on the input uncertainties $\{x\}$. The most straightforward solution strategy may seem to be applying a full interval translation of the stiffness matrix assembly and a subsequent linear solver using interval arithmetics. However, this approach presents a major drawback: it generally results in a very high overestimation of the exact results, caused by an

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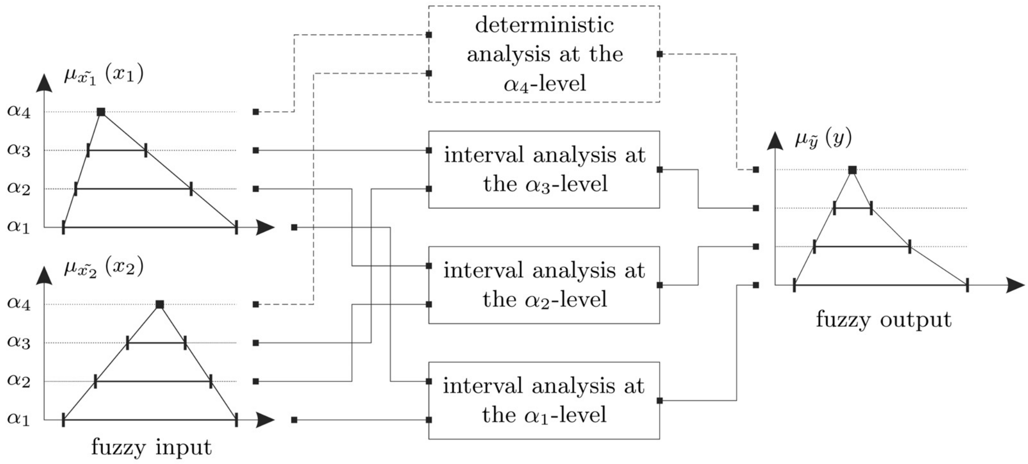


Fig. 1. Fuzzy FEM.

accumulation of conservatism with each interval operation [6]. The conservatism is due to the fact that the different interval arithmetic-based methods cannot keep track of the possible dependencies between different operands. For example, suppose we obtain the interval stiffness matrix [***K***]. In a solution strategy for interval linear systems, the elements of [***K***] are considered as independent, and thus the correlation between them through the uncertain parameters is lost. In order to reduce this conservatism, this paper introduces a static IFEM procedure based on the hybrid approach [1]. This method is compared with a global optimisation approach and a vertex analysis. All methods are briefly reviewed. The presented techniques are illustrated using a numerical analysis in Section 3.

2.1. Vertex method

Vertex analysis is the most obvious approach to obtain interval displacements [7]. The procedure consists of taking the minimum and the maximum of the results obtained on the 2^{n_u} vertices of the uncertain parameter space (n_u – the number of uncertain parameters). To obtain a full set of interval displacements, 2^{n_u} crisp system solutions are needed. However, the method has an important limitation: the exact interval results can be obtained only if the behaviour of the displacement in function of the input parameter space is monotonic, which is difficult to verify in a general case.

2.2. Global optimisation approach

This approach computes an interval result of the displacement component u_d through a global optimisation. The objective function is taken as the objective function, which is minimised and maximised over the complete uncertainty input parameter space in order to obtain the interval result:

$$\boldsymbol{u}(x)_d = \left[\min_{\{x\}\in\{\boldsymbol{x}\}} (u(x)_d);\ \max_{\{x\}\in\{\boldsymbol{x}\}} (u(x)_d)\right] \quad (2)$$

This approach gives the exact interval results.

2.3 Hybrid approach

In order to extend the applicability of IFEM, a general remedy to the excessive conservatism in the interval arithmetic approach was introduced by Moens and Vandepitte [5]. It is a hybrid procedure, consisting of both a global optimisation and an interval arithmetic part. In the first part, an optimisation is applied to calculate the interval result at some intermediate step of the total algorithm. In the second part, the interval analysis is performed on these intermediate results. This method has two major advantages:

- because of the global optimisation, all conservatism prior to the optimised intermediate result is neutralised;
- the performance of the optimisation step is controllable by adequately choosing the level on which to perform it.

An overview of the application of the hybrid approach for static IFE analysis together with the global optimisation approach is presented in Fig. 2. In the hybrid approach, the desired displacement component is obtained in two steps. In the first step, the interval inverse of the corresponding elements of the stiffness matrix ($[\boldsymbol{K}]_{d,l}$) are obtained through optimisation. In the second step of the procedure, the interval displacement vector {***u***} is obtained with interval arithmetics (see Fig. 2). This hybrid method overestimates the displacements somewhat due to conservatism in the second step.

Hybrid approach	Optimisation of the different elements of $[K]^{-1}_{d,l}$	$[K(x)]^{-1}$ $k(x)^{-1}_{i,j} = [K(x)]^{-1}_{i,j}$ $\boldsymbol{k}(x)^{-1}_{d,l}\mid_{l=1\ldots nf} = \left[\min_{\{x\}\in\{\boldsymbol{x}\}}(k(x)^{-1}_{d,l});\ \max_{\{x\}\in\{\boldsymbol{x}\}}(k(x)^{-1}_{d,l})\right]$	$[K]$-the stiffness matrix $\{K\}_d$-the d^{th} row of $[K]$ $\{F\}$-the load vector n-the dimension of the problem
	+ interval arithmetics	$\{\boldsymbol{u}_d\} = \{\boldsymbol{K}^{-1}\}_d \cdot \{F\}$	d-the index of the desired displacement component
Global optimisation approach	Optimisation of the displacement u_d	$\{u(x)\} = [K(x)]^{-1} \cdot \{F\}$ $u(x)_d = \sum_{l=1}^{nf} k(x)^{-1}_{d,l} \cdot f_l$ $\boldsymbol{u}(x)_d = \left[\min_{\{x\}\in\{\boldsymbol{x}\}}(u(x)_d);\ \max_{\{x\}\in\{\boldsymbol{x}\}}(u(x)_d)\right]$	$\{x\}$-the uncertain parameters $i, j, l \leq n$-indices nf-number of nonzero elements in the lefthandside $\{F\}$

Fig. 2. Hybrid and global optimisation approaches.

The computational performance of the global optimisation and the hybrid approaches is determined by the optimisation. The optimisation depends on two important issues: the smoothness of the objective function in the domain of the input parameter space and the computational cost of the evaluation of the objective function. The objective function in both approaches involves the solution of a linear system. Therefore, the relative computational performances of these approaches are determined mainly by the smoothness of the objective function, which is strongly problem-and uncertainty-dependent and often difficult to predict. For certain parameters (e.g. the material Young's modulus), the dependency between the parameter and the objective functions is clearly linear. For other parameters, for instance geometrical dimensions, which affect the Jacobian matrix of an isoparametric element, the dependency is not so clear.

3. Numerical example

The strategies discussed are demonstrated through the example shown in Fig. 3. The problem is a simple two-dimensional model, discretised using plane stress elements. The structure is subject to clamping and three different nodal loads. The uncertain parameters are the length of the structure (nominal 0.3 m), the thickness of elements (nominal 3 mm), and the Young's modulus (nominal 210 MPa) for element 4. The range of the uncertain parameters is taken to be the nominal value ± 15%. An extra restriction was imposed on the thickness uncertainty, so that the mean thickness $\frac{\sum_{i=1}^{n_{el}} t}{n_{el}} = 3$ mm.

Fig. 4(a) shows the interval translation (dotted rectangle) results versus MC results of 1000 uniformly distributed samples over the uncertainty parameters. The order of overestimation of the interval arithmetic approach is enormous. Fig. 4(b) shows the MC samples compared with the other implementations. The global

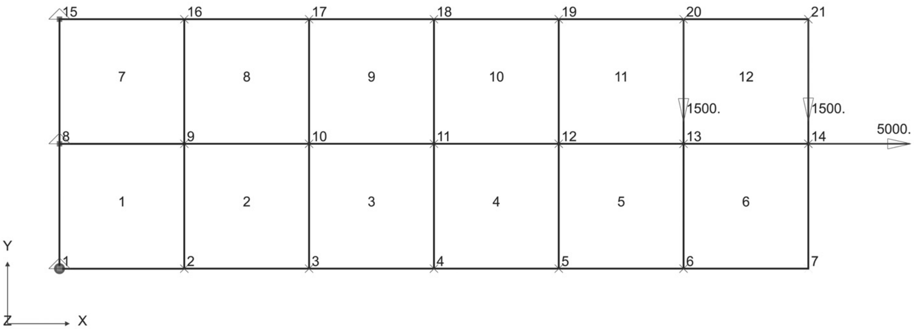
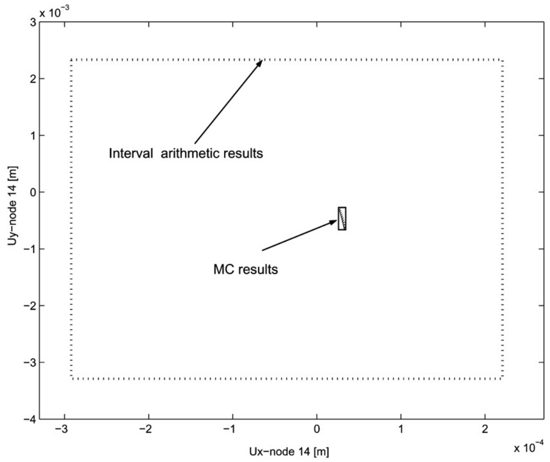
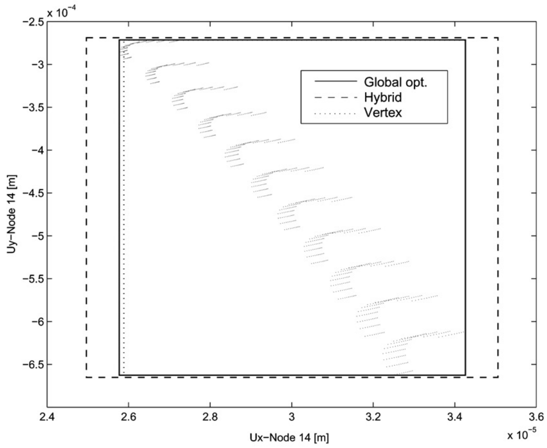


Fig. 3. Numerical example.



(a)



(b)

Fig. 4. Numerical results. (a) Interval arithmetics vs MC. (b) Other approaches vs MC.

optimisation approach result is a closely circumscribing rectangle around the MC samples. The dashed rectangle shows the bounds of the hybrid results. A slight amount of conservatism due to interval arithmetics can be observed. The dotted rectangle shows the non-conservative results of a vertex analysis. In this case, the vertex result gives a good approximation, since the global optima lie very close to the vertex solutions.

4. Conclusions and acknowledgement

This paper focuses on different approaches for IFEM in static analysis. These could serve as an alternative for the full interval translation approach, which is subject to an extremely high amount of conservatism. Three alternative approaches were presented and tested: the vertex method, a global optimisation approach and a hybrid approach. Of all presented approaches, the vertex method is clearly computationally the least expensive. However, in order for the vertex result to be correct, the analysed displacement should have a monotonic behaviour with respect to the uncertain input parameters. If not, then the vertex results are non-conservative and, therefore, of little value in a design context. Theoretically, the global optimisation approach always gives the correct interval results. The computational cost of this approach is rather unpredictable. When the uncertain parameters have a strongly nonlinear influence on the computed displacements, the optimisation can become very costly or even fail. The hybrid approach is most appropriate for problems where the influence of the uncertain parameters on the terms in the stiffness matrix is known to be monotonic. The results of this approach will always be conservative due to the interval arithmetic part of the procedure.

In a future work, the global optimisation approach will be improved in a reduced optimisation. The reduced optimisation technique is based on a vertex analysis. Further, a response surface approximation [8] seems appropriate to significantly reduce the computational cost of this approach. Other future objectives are to develop a technique to compute a reasonably conservative stress field and to extend the linear static problem to include uncertainties in the essential and natural boundary conditions, which are affecting the right-hand side of the linear static equation.

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