Softening cohesive interface analysis via boundary integral equations

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Abstract

An incremental symmetric boundary integral formulation for the problem of bodies connected by non-linear cohesive interface is here presented. The numerical solution example of the incremental problem is achieved by the symmetric Galerkin boundary element method (SGBEM). Numerical results of FRP-concrete delamination failure, obtained by coupling the incremental SGBEM system with a local arc-length constraint, are presented and compared with the FEM solution.

Keywords: Boundary integral equations; Cohesive interface; Arc-length; SGBEM

1. Introduction

The cohesive forces acting at the interface between bodies are actually one of the most important constitutive parameters, determining the strength and stability of structures. For example, displacement-softening interface responses often imply a global strain-softening structural response. In the calculation process, to follow the quasi-static equilibrium path beyond the onset of snap-trough or snap-back, the basic idea for a flexible incrementation control technique is that the step is specified by a constraint equation, which involves both the problem unknowns and the load multiplier. Since failure is localized along the interfaces, global constraint equations including all the problem unknowns seem to be redundant to produce a converging solution, essentially because they involve unknowns that are not responsible for the equilibrium instabilities [1].

Boundary integral equations (BIEs) are very attractive for such problem, because all non-linearities are localized on the boundary of assumed linear elastic domains. The integral operator that governs the problem, as was formulated in [2], is proved to be linear with respect to the rate unknown fields and symmetric with respect to a classical bilinear form in the presence of a holonomic interface law. The numerical solution of the incremental problem is then achievable by the SGBEM.

The enlarged system of equations, obtained coupling the incremental SGBEM system equations with a local arc-length constraint, becomes singular only at bifurcation points [3]. Numerical examples of FRP-concrete delamination failure, obtained with this technique, are presented and compared with the FEM solution.

2. Boundary integral formulation

We begin this section with a very brief review of BIEs for elasticity, their approximation via the symmetric-Galerkin procedure. The reader is asked to consult the cited references for further details.

Consider, in a right-hand Cartesian reference system $(x_1, x_2)^t \equiv \mathbf{x}$, an elastic homogeneous and isotropic material occupying the finite simply connected domain $\Omega \subset \mathbb{R}^2$, with a piecewise smooth boundary $\Gamma = \Gamma_u \cup \Gamma_p$, where Γ_u, Γ_p are open disjoint subsets of Γ ($\Gamma_u \cap \Gamma_p = \emptyset$). Assuming small strains and displacements, consider, in absence of body force, its response to quasi-static external actions: traction $\bar{\mathbf{p}}$ on Γ_p, displacements $\bar{\mathbf{u}}$ on Γ_u. For this problem we may derive for the Cauchy data $(\mathbf{u}, \mathbf{p})^t$ a system of BIEs:

$$\frac{1}{2}\begin{pmatrix}\mathbf{u}\\ \mathbf{p}\end{pmatrix} = \begin{pmatrix}-K & V\\ D & K'\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{p}\end{pmatrix}, \qquad \mathbf{x} \in \Gamma \tag{1}$$

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using the single-layer potential

$$(V\mathbf{p})_\ell(\mathbf{x}) = \int_\Gamma U_{\ell m}(\mathbf{x},\ \mathbf{y})p_m(\mathbf{y})\ \mathrm{d}s_\mathbf{y},$$
$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{2}$$

the double-layer potential

$$(K\mathbf{u})_\ell(\mathbf{x}) = \int_\Gamma T_\mathbf{y} U_{\ell m}(\mathbf{x},\ \mathbf{y})u_m(\mathbf{y})\ \mathrm{d}s_\mathbf{y},$$
$$K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{3}$$

and the hypersingular integral operator

$$(D\mathbf{u})_\ell(\mathbf{x}) = T_\mathbf{x}\int_\Gamma T_\mathbf{y} U_{\ell m}(\mathbf{x},\ \mathbf{y})u_m(\mathbf{y})\ \mathrm{d}s_\mathbf{y},$$
$$D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \tag{4}$$

K is defined by Cauchy singular integrals and D is defined by a hypersingular finite part integral in the sense of Hadamard (see [4]). In Eq. (1) the operator K' is the adjoint of K with respect to the natural duality $<.,.>$ between the Sobolev space $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$ [5]. The definition of all these boundary potentials is based on a fundamental solution, which is given by the Kelvin solution $U_{m\ell}$ $(\mathbf{x},\ \mathbf{y})$ [3].

In Eqs (3) and (4) $T_\mathbf{y}$ denotes the traction operator on Γ with differentiations with respect to $\mathbf{y}$ and $T_\mathbf{y}U(\mathbf{x},\ \mathbf{y})$ is the boundary stress tensor of the fundamental solution. If we write the first equation on Γ_u and the second one on Γ_p, inserting boundary data: $\bar{\mathbf{u}}$ on Γ_u, $\bar{\mathbf{p}}$ on Γ_p, we obtain a system of two BIEs of first kind for the unknown Cauchy data $\mathbf{p}$ on Γ_u and $\mathbf{u}$ on Γ_p, of the form:

$$\begin{pmatrix} V_{uu} & -K_{pu} \\ K'_{up} & D_{pp} \end{pmatrix}\begin{pmatrix} \mathbf{p} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} -V_{pu} & \frac{1}{2}I + K_{uu} \\ \frac{1}{2}I - K'_{pp} & D_{up} \end{pmatrix}\begin{pmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{u}} \end{pmatrix} \tag{5}$$

where the boundary integral operators subscripts ab mean integration over Γ_a and evaluation over Γ_b, with $a,\ b\ =\ u,\ p$.

Now we consider two-dimensional homogeneous elastic bodies occupying the domains Ω_1 and Ω_2, bounded by exterior boundaries Γ^1, Γ^2 with outward unit normal $\mathbf{n}^1$ and $\mathbf{n}^2$, respectively, and connected by an interface $\Gamma_i\ =\ \Gamma^1 \cap \Gamma^2$ (see Fig. 1). Quasi static external tractions $\bar{\mathbf{p}}_j(t,\ \mathbf{x})$ are imposed on the Neumann boundary Γ_p^j and displacements $\bar{\mathbf{u}}_j(t,\ \mathbf{x})$ on the Dirichlet boundary Γ_u^j of each domain Ω_j $(\Gamma_u^j \cup \Gamma_p^j = \Gamma^j\backslash\Gamma_i;\ \Gamma_u^j \cap \Gamma_p^j = \emptyset)$, $j = 1, 2$. All are given functions of the *time-like* parameter t. Small displacements and strains hypothesis implies: $\mathbf{n}^1 \doteq \mathbf{n}(\mathbf{x}^1)\ =\ -\mathbf{n}(\mathbf{x}^2) \doteq -\mathbf{n}^2$ with $\mathbf{x}^j \in \Gamma_i^j$, where we denoted Γ_i^j the outline Γ_i in Ω^j. Moreover the following assumptions are adopted herein: the known a priori interface Γ_i is the locus of possible displacement

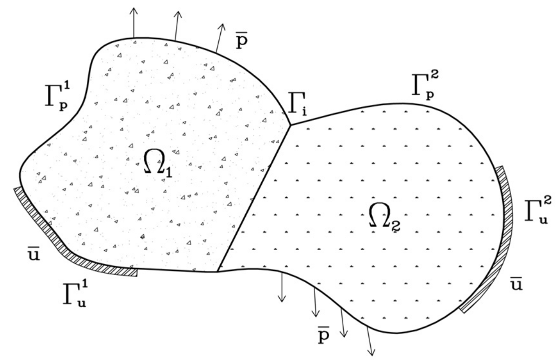

Fig. 1. Two domains connected by a non-linear cohesive interface.

discontinuites: $\mathbf{w}(t,\mathbf{x})\ =\ \mathbf{u}_1(\mathbf{x}_1(t))\ -\ \mathbf{u}_2(\mathbf{x}_2(t)),\ \mathbf{x}_j(t) \in \Gamma_i^j,\ j = 1, 2$; equilibrate tractions act: $\mathbf{p}_1\ (t,\ \mathbf{x})\ =\ -\mathbf{p}_2(t,\ \mathbf{x})$ and $\mathbf{n}(\mathbf{x})\ =\ \mathbf{n}^1(\mathbf{x})$.

The incremental form of Eq. (3) on Dirichlet and Neumann boundaries on the jth domain reads:

$$V_{uu}^j\dot{\mathbf{p}}_j - K_{pu}^j\dot{\mathbf{u}}_j + V_{iu}^j\dot{\mathbf{p}}_j - K_{iu}^j\dot{\mathbf{u}}_j = \frac{1}{2}\dot{\bar{\mathbf{u}}}_j - V_{pu}^j\dot{\bar{\mathbf{p}}}_j +$$
$$K_{uu}^j\dot{\bar{\mathbf{u}}}_j \doteq \mathbf{f}_u^j \qquad \mathbf{x} \in \Gamma_u^j, \quad j = 1, 2 \tag{6}$$

$$-K'^j_{up}\dot{\mathbf{p}}_j + D_{pp}^j\dot{\mathbf{u}}_j - K'^j_{ip}\dot{\mathbf{p}}_j + D_{ip}^j\dot{\mathbf{u}}_j = -\frac{1}{2}\dot{\bar{\mathbf{p}}}_j + K'^j_{pp}\dot{\bar{\mathbf{p}}}_j -$$
$$D_{up}^j\dot{\bar{\mathbf{u}}}_j \doteq \mathbf{f}_p^j \qquad \mathbf{x} \in \Gamma_p^j, \quad j = 1, 2 \tag{7}$$

The overhead dots denote the derivative with respect to time-like parameter t. The previous equations (displacement and traction BIEs) are written for the two faces of the interface:

$$-\frac{1}{2}\dot{\mathbf{u}}_j + V_{ui}^j\dot{\mathbf{p}}_j - K_{pi}^j\dot{\mathbf{u}}_j + V_{ii}^j\dot{\mathbf{p}}_j - K_{ii}^j\dot{\mathbf{u}}_j = -V_{pi}^j\dot{\bar{\mathbf{p}}}_j +$$
$$K_{ui}^j\dot{\bar{\mathbf{u}}}_j \doteq \mathbf{f}_{i(u)}^j \qquad \mathbf{x} \in \Gamma_{i(u)}^j, \quad j = 1, 2 \tag{8}$$

$$\frac{1}{2}\dot{\mathbf{p}}_j + K'^j_{ui}\dot{\mathbf{p}}_j + D_{pi}^j\dot{\mathbf{u}} - K'^j_{ii}\dot{\mathbf{p}}_j + D_{ii}^j\dot{\mathbf{u}}_j = K'^j_{pi}\dot{\bar{\mathbf{p}}}_j -$$
$$D_{ui}^j\dot{\bar{\mathbf{u}}}_j \doteq \mathbf{f}_{i(p)}^j \qquad \mathbf{x} \in \Gamma_{i(p)}^j, \quad j = 1, 2 \tag{9}$$

As a further assumption, traction $\mathbf{p}$ and relative opening displacement $\mathbf{w}$ are related by non-linear cohesive law $\mathbf{p}(\mathbf{w}(t,\mathbf{x}))$, for all $\mathbf{x}\ \in\ \Gamma_i$. The interface constitutive equation can be written in an incremental form making use of the tangent matrix of the cohesive law, denoted with $\mathcal{D}_T$:

$$\dot{\mathbf{p}}(t, \mathbf{x}) = \mathcal{D}_T(\mathbf{w}(t, \mathbf{x}))\dot{\mathbf{w}}(t, \mathbf{x}) \tag{10}$$

Equations (8) and (9) are related together in order to achieve a symmetric formulation. In fact a direct exploitation of these equations will not lead to a symmetric formulation. For this reason we define two new vectors on the interface: the mean displacement $\dot{\mathbf{v}}$ and the half opening displacement $\dot{\mathbf{z}}$:

$$\dot{\mathbf{v}}(t,\mathbf{x}) = \frac{1}{2}(\dot{\mathbf{u}}_1(\mathbf{x}(t)) + \dot{\mathbf{u}}_2(\mathbf{x}(t))),$$

$$\dot{\mathbf{z}}(t,\mathbf{x}) = \frac{1}{2}(\dot{\mathbf{u}}_1(\mathbf{x}(t)) - \dot{\mathbf{u}}_2(\mathbf{x}(t))) = \frac{1}{2}\dot{\mathbf{w}}(t,\mathbf{x}) \quad (11)$$

BIEs written for each domain can be put together for use in an overall analysis by employing the conditions of equilibrium of traction components and incremental cohesive opening traction-displacement relationship. Now combining Eqs (8) and (9) on the interface Γ_i and having set $\dot{\mathbf{p}}\ (t,\ \mathbf{x}) = \mathcal{D}_T\ (\mathbf{w}(t,\mathbf{x}))\dot{\mathbf{w}}(t,\mathbf{x}) = \hat{\mathcal{D}}_T(\mathbf{z}(t,\mathbf{x}))\dot{\mathbf{z}}(t,\mathbf{x})$, we obtain the following system of BIEs (see [3]):

$$V^1_{uu}\dot{\mathbf{p}}_1 - K^1_{pu}\dot{\mathbf{u}}_1 - K^1_{iu}\dot{\mathbf{v}} + (V^1_{iu}\hat{\mathcal{D}}_T - K^1_{iu})\dot{\mathbf{z}} = \mathbf{f}^1_u \quad \mathbf{x} \in \Gamma^1_u$$

$$-K'^1_{up}\dot{\mathbf{p}}_1 + D^1_{pp}\dot{\mathbf{u}}_1 + D^1_{ip}\dot{\mathbf{v}} - (K'^1_{ip}\hat{\mathcal{D}}_T - D^1_{ip})\dot{\mathbf{z}} = \mathbf{f}^1_p \quad \mathbf{x} \in \Gamma^1_p$$

$$V^2_{uu}\dot{\mathbf{p}}_2 - K^2_{pu}\dot{\mathbf{u}}_2 - K^2_{iu}\dot{\mathbf{v}} + (V^2_{iu}\hat{\mathcal{D}}_T - K^2_{iu})\dot{\mathbf{z}} = \mathbf{f}^2_u \quad \mathbf{x} \in \Gamma^2_u$$

$$-K'^2_{up}\dot{\mathbf{p}}_2 + D^2_{pp}\dot{\mathbf{u}}_2 + D^1_{ip}\dot{\mathbf{v}} - (K'^2_{ip}\hat{\mathcal{D}}_T - D^2_{ip})\dot{\mathbf{z}} = \mathbf{f}^2_p \quad \mathbf{x} \in \Gamma^2_p$$

$$-K'^1_{ui}\dot{\mathbf{p}}_1 + D^1_{pi}\dot{\mathbf{u}}_1 + K'^2_{ui}\dot{\mathbf{p}}_2 - D^2_{pi}\dot{\mathbf{u}}_2 + (D^1_{ii} + D^2_{ii})\dot{\mathbf{v}} + -(K'^1_{ii}\hat{\mathcal{D}}_T - D^1_{ii} - K'^2_{ii}\hat{\mathcal{D}}_T + D^2_{ii})\dot{\mathbf{z}} = \mathbf{f}^1_{i(p)} - \mathbf{f}^2_{i(p)} \quad (12)$$

$$\mathbf{x} \in \Gamma_i$$

$$(\hat{\mathcal{D}}^t_T V^1_{ui} - K'^1_{ui})\dot{\mathbf{p}}_1 - (\hat{\mathcal{D}}^t_T V^2_{ui} - D^1_{pi})\dot{\mathbf{u}}_1 - (\hat{\mathcal{D}}^t_T V^2_{ui} + K'^2_{ui})\dot{\mathbf{p}}_2 + (\hat{\mathcal{D}}^t_T K^2_{pi} + D^2_{pi})\dot{\mathbf{u}}_2 + -(\hat{\mathcal{D}}^t_T - D^1_{ii} + \hat{\mathcal{D}}^t_T K^2_{ii} - D^2_{ii})\dot{\mathbf{v}} + \Big[(\hat{\mathcal{D}}_T - \hat{\mathcal{D}}^t_T) + (\hat{\mathcal{D}}^t_T V^1_{ii}\hat{\mathcal{D}}_T - \hat{\mathcal{D}}^t_T K^1_{ii} - K'^1_{ii}\hat{\mathcal{D}}_T + D^1_{ii}) + (\hat{\mathcal{D}}^t_T V^2_{ii}\hat{\mathcal{D}}_T - \hat{\mathcal{D}}^t_T K^t_{ii} - K'^2_{ii}\hat{\mathcal{D}}_T + D^2_{ii})\Big]\dot{\mathbf{z}} = \hat{\mathcal{D}}^t_T(\mathbf{f}^1_{i(u)} - \mathbf{f}^2_{i(u)}) + \mathbf{f}^1_{i(p)} - \mathbf{f}^2_{i(p)} \quad \mathbf{x} \in \Gamma_i$$

The boundary integral equations (12) can be expressed in the condensed form:

$$\mathcal{N}(\zeta(t))\dot{\zeta}(t) = \mathbf{F}(\zeta(t)) \quad (13)$$

The unknown vector $\dot{\zeta}$ is made of tractions $\dot{\mathbf{p}}$ on Dirichlet boundaries Γ^j_u, displacements $\dot{\mathbf{u}}$ on Neumann boundaries Γ^j_p, mean displacements $\dot{\mathbf{v}}$ and relative half opening displacements $\dot{\mathbf{z}}$ on the cohesive interface Γ_i. The following variational statement holds.

Proposition [2] If the tangent matrix of the cohesive law $\mathcal{D}_T$ is symmetric, i.e. $\mathcal{D}_T = \mathcal{D}^t_T$, the integral operator $\mathcal{N}$: $\mathbb{H} \rightarrow \mathbb{K}$ is symmetric with respect to the bilinear form:

$$\mathcal{B}(h', k') \doteq \int_\Gamma h' \cdot k' \,\mathrm{d}s \qquad h' \in \mathbb{H},\ k' \in \mathbb{K} \quad (14)$$

3. Numerical solution

Consider the non-linear incremental problem (13) where the structural response ζ is obtained for $\bar{\mathbf{p}}$ varying from a null initial value and evolving quasi statically in time by means of a load factor $\lambda(t)$, i.e. $\mathbf{F}(\zeta(t)) = \dot{\lambda}(t)\mathbf{f}(\zeta(t))$, supposing that λ at least initially increases. The problem under consideration may present limit points, and conventional incremental algorithms, based on a fixed value of load increment, may fail to overcome such points. Hence, path-following techniques must be used whereby the load factor λ is added to the set of unknowns. Problem (14) becomes therefore:

$$\mathcal{N}(\zeta(t))\dot{\zeta}(t) = \dot{\lambda}(t)\mathbf{f}(\zeta(t)) \quad (15)$$

The arc-length method is based on the introduction of a control function giving a measure of the evolution of the loading process. The form of the constraint equation, proposed originally in [6], reads:

$$\|\dot{\zeta}\|^2 + \dot{\lambda}^2 = c(\lambda, \zeta)^2 \quad (16)$$

where c represents the arc-length of the equilibrium path. In the present paper, a local control function, analogous to that proposed in [7], is used. Differently from other papers, the present formulation is differential in time, whereby the solution is reached by (explicit) time integration strategies (see [3]).

The proposed model has been used to simulate some FRP – concrete delamination test. Figure 2 shows a typical pull–pull delamination test. Left and bottom sides have been constrained in order to have no displacements in the direction normal to the surface and free displacements tangent to it with reference to this setup. The behaviour of the bond between concrete and

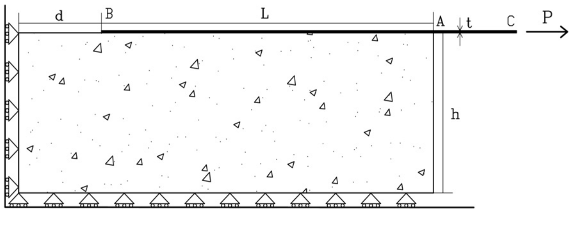

Fig. 2. FRP-concrete delamination test.

reinforcement can be characterized by a relationship between shear stress locally transferred between concrete and reinforcement to slip [8] with a linear elastic branch until the maximum shear stress value is reached and by a linear softening branch up to delamination:

$$p = \begin{cases} k\,w & \text{if} w \le w_0 \\ \frac{w_1 - w}{w_1 - w_0}\,\bar{p}_2 & \text{if} w_0 \le w \le w_1 \\ 0 & \text{if} w \ge w_1 \end{cases} \quad (17)$$

where w_0, w_1 and $\bar{p}_2$ are given quantities.

The geometrical, mechanical properties of materials and interface parameters are reported in [9]. In Fig. 3 load-displacement curves obtained via BEM and via FEM are reported for three different points: u_A, u_B, u_C (see Fig. 2). The methods, whose discretizations are presented in Fig. 4, give practically the same results. In Table 1 are reported discretization data and the number of matrix evaluations. We adopted an explicit second-order scheme, otherwise in the FEM context an incremental-iterative scheme is used [1].

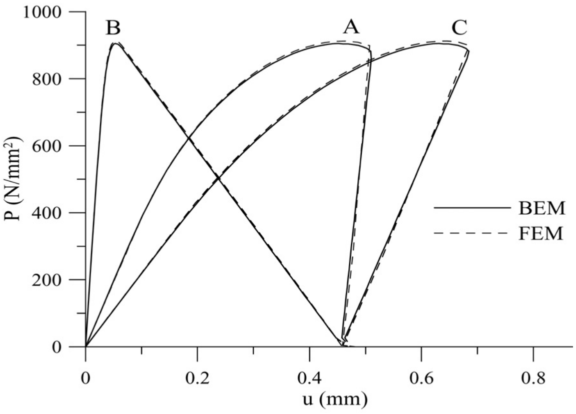

Fig. 3. BEM and FEM solutions.

Table 1
Discretization data for FRP delamination test

	Nodes	Elements	Degrees of freedom	Interface nodes	Matrix evaluation
BEM	219	220	643	101	144
FEM	1830	1881	3680	101	163

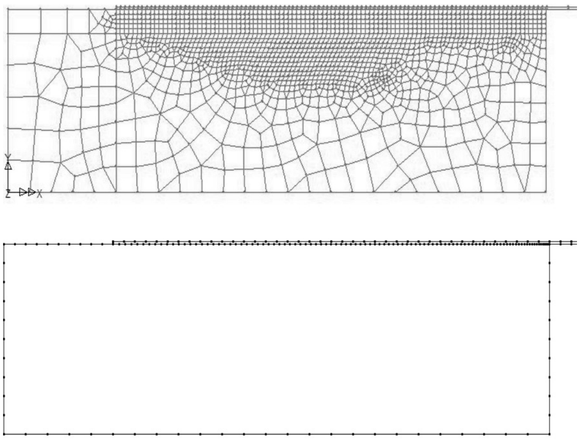

Fig. 4. FEM and BEM discretizations.

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