Some solutions to the asymptotic bending problem of non-inhibited shells

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Abstract

This paper studies the asymptotic limit bending problem of thin linearly elastic shells as the thickness goes to zero. Such asymptotic bending problem makes sense whenever bending problems are admissible. Two alternate expressions of the change of curvature tensor are presented, giving new formulations for the asymptotic bending problem. In the case of cylinders and hyperbolic surfaces, we present some solutions, including analytic solutions. Such solutions can constitute tests for membrane locking.

Keywords: Non-inhibited shell; Asymptotic bending problem; Inextensional displacement; Membrane locking

1. Introduction

The natural trend for a very thin elastic shell is to perform bendings as shown by the asymptotic analysis of linear thin shells. Unlike the bendings in beams problems, bending deformation on a surface is not always admissible: it depends on the boundary conditions and the geometry [1]. Thin shells are thus classified as [with bendings] *inhibited* or *not-inhibited* [2,3]. This classification is also referred to as *membrane dominated* and *bending dominated* [4]. These two very distinct asymptotic behaviors lead to strong difficulties in the numerical finite element approximation of very thin elastic shells: boundary layers, propagation and reflexion of singularities, sensitivity within the inhibited case and numerical (membrane) locking in the non-inhibited or bending dominated case, see [2,3,5,6,7,8].

In this paper, the limit problem of thin non-inhibited elastic shells, the asymptotic bending problem, is studied, making sense whenever the set of kinematically admissible infinitesimal bending is not reduced to trivial bendings (rigid displacements).

With alternative formulations, the asymptotic bending problem simplifies greatly in some cases of cylinder and hyperbolic shells and allows an easy analytical solution, eventually.

Such solution can be used as a test, as in [9], to the membrane locking that appears in the finite element approximation of very thin elastic shell problems [3,5,7,10].

2. The asymptotic bending problem

In this paper we will employ the Einstein convention of summation on repeated upper or lower indices. The Greek (resp. Latin) indices range over {1,2} (resp. {1,2,3}). The partial derivatives with respect to variables x_α are denoted in lower indices preceded by a comma. An overarrow is used to indicate space vectors. The variables x_1, x_2 live in the bounded domain $\Omega \subset \mathbb{R}^2$.

Let S be a surface given by a map $(\Omega, \vec{r})$, $\Omega \in \mathbb{R}^2$ with

$$\vec{r} = x(x_1, x_2)\vec{e}_1 + y(x_1, x_2)\vec{e}_2 + z(x_1, x_2)\vec{e}_3 \quad (1)$$

The vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ constitute a Cartesian basis of space R^3 and the direction $\vec{e}_3$ will be referred to as the vertical direction. The coefficients of the first and second fundamental form are

$$a_{\alpha\beta} = \vec{a}_\alpha . \vec{a}_\beta \quad \text{and} \quad b_{\alpha\beta} = \vec{a}_3 . \vec{a}_{\alpha,\beta} \quad (2)$$

where the tangent vectors $\vec{a}_\alpha = \vec{r}_{,\alpha}$ and the unit normal vector $\vec{a}_3$ constitute the *covariant basis* on S. The covariant basis is associated with its dual basis: the contravariant basis $(\vec{a}^1, \vec{a}^2, \vec{a}^3)$. We also have the coefficients $a^{\alpha\beta} = \vec{a}^\alpha\, \vec{a}^\beta$ defining the inverse matrix to $(a_{\alpha\beta})$. The Christoffel's symbols are

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$$\Gamma^{\mu}_{\alpha\beta} = \vec{a}^{\mu}.\vec{a}_{\alpha,\beta} \tag{3}$$

We let ε denote the thickness so that the shell, the midsurface of which is S, is defined as the set $\{\vec{r}(x_1, x_2) + x_3\vec{a}_3, x_3 \in [-\varepsilon/2, \varepsilon/2]\} \subset \mathbb{R}^3$

Let us consider a shell problem. The mechanical problem in the space of kinematically admissible displacements $\vec{V}$ and with external applied load L^{ϵ}, writes

$$\begin{cases} \text{find } \vec{u}^{\varepsilon} \in \vec{V} \text{ such that} \\ {}^{a}\varepsilon(\vec{u}^{\varepsilon}, \vec{v}) = L^{\varepsilon}(\vec{v}), \quad \forall \vec{v} \in \vec{V} \end{cases} \tag{4}$$

where ${}^{a}\varepsilon$ denotes the deformation energy bilinear form which depends on the linearized variations of the fundamental forms of the surfaces, especially the symmetric tensors $\rho_{\alpha\beta}$ and $\gamma_{\alpha\beta}$, respectively the change of curvature tensor and the membrane strain tensor. $\rho_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ are the linearized variations of the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$.

Classically, we have the following asymptotic behavior valid for thin shells linear models such as Koiter's or Naghdi's [2,3,4]:

Theorem 2.1 *If the (scaled) load $L^{\varepsilon}(\vec{v})$ depends on ε as $L^{\varepsilon}(\vec{v}) = \varepsilon^3 L(\vec{v})$, then the solutions $\vec{u}^{\varepsilon}$ of Eq.* (4) *converge towards the solutions $\vec{u}^0$ of the asymptotic limit bending problem*

$$\begin{cases} \textit{find } \vec{u}^0 \in \vec{G} \textit{ such that} \\ \int_S \frac{A^{\alpha\beta\lambda\mu}}{12} \rho_{\alpha\beta}(\vec{u})\rho_{\lambda\mu}(\vec{v}), = L(\vec{v}), \quad \forall \vec{v}^* \in \vec{G} \end{cases} \tag{5}$$

where $\vec{G}$ is the set of kinematically admissible infinitesimal bendings (also called inextensional displacement). In the case of homogeneous and isotropic material,

$$A^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\alpha\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu} \right] \tag{6}$$

where E is the Young's modulus and ν the Poisson ratio.

3. Resolution of the asymptotic bending problem

The limit bending problem in Eq. (5) is a constrained problem, thus one can try to solve it with a mixed formulation. However, at least two alternate formulations of the asymptotic bending problem, leading to important simplifications, permit its resolution in specific cases. The first is based on the notion of infinitesimal rotation field associated with an inextensional displacement and has been developed in [9] exhibiting some solutions for hyperbolic surfaces. The second is based on a new expression of the tensor $\rho_{\alpha\beta}$ with only one component of the displacement, in the case of an inextensional bending. Both methods take advantage of properties of infinitesimal bendings, for which we recall some properties.

The infinitesimal bendings or inextensional displacements are displacements leaving the linearized variation of the first fundamental form, that is the intrinsic metric, unchanged. This variation is given by the membrane strain tensor

$$\gamma_{\alpha\beta}(\vec{u}) = \frac{1}{2}(\vec{u}_{,\alpha}.\vec{r}_{,\beta} + \vec{r}_{,\alpha}.\vec{u}_{,\beta}) \tag{7}$$

Definition 3.1. *A displacement $\vec{u}$ is an inextensional displacement if $\gamma_{\alpha\beta}(\vec{u}) = 0$. Any inextensional displacement $\vec{u}$ defines a unique vector field $\vec{\omega}$, called associated infinitestimal rotation field satisfying the relations:*

$$\vec{u}_{,1} = \vec{\omega} \wedge \vec{a}_1 \quad \textit{and} \quad \vec{u}_{,2} = \vec{\omega} \wedge \vec{a}_2 \tag{8}$$

Let us introduce the notation $\vec{w}_{\alpha} = w^{\mu}_{\alpha}\vec{a}_{\mu}$ for the partial derivatives of a rotation field $\vec{\omega}$:

$$\vec{w}_{\alpha} = \vec{\omega}_{,\alpha} \tag{9}$$

It has been shown that such vector fields are tangent to the surface and their components satisfy a partial differential system, the characteristics of which coincide with the asymptotic lines of the surface [9]:

$$w^1_{1,2} + \Gamma^1_{2\lambda}w^{\lambda}_1 = w^1_{2,1} + \Gamma^1_{1\lambda}w^{\lambda}_2 \tag{10}$$

$$w^2_{1,2} + \Gamma^2_{2\lambda}w^{\lambda}_1 = w^2_{2,1} + \Gamma^2_{1\lambda}w^{\lambda}_2 \tag{11}$$

$$b_{12}w^1_1 + b_{22}w^2_1 = b_{11}w^1_2 + b_{12}w^2_2 \tag{12}$$

$$w^1_1 + w^2_2 = 0 \tag{13}$$

3.1. The tensor of curvature variation for inextensional displacements

We give here two alternative expressions for the tensor of curvature variation in the case of inextensional displacements. One is based on the associated rotation field, and the second is based on only one cartesian component of the displacement.

Proposition 3.2. *Let $\vec{u}$ be an inextensional displacement on the considered surface S and let $\rho_{\alpha\beta}(\vec{u})$ be the tensor of curvature variation. If we denote by $\omega^{\lambda}_{\alpha}$ the contravariant components of $(\vec{w}_1, \vec{w}_2)$, the partial derivatives of the associated rotation field $\vec{\omega}$, we have:*

$$\begin{cases} \rho_{11}(\vec{u}) & = & -w^2_1\sqrt{a} \\ \rho_{22}(\vec{u}) & = & +w^1_2\sqrt{a} \\ \rho_{12}(\vec{u}) & = & \frac{1}{2}(w^1_1 - w^2_2)\sqrt{a} \end{cases} \tag{14}$$

Obtaining Eq. (14) is quite straightforward writing the variation of the coefficients of the second fundamental form $b_{\alpha\beta}$ and replacing the derivatives of an inextensional displacement with its associated rotation field.

Another expression is based on the possibility to reconstruct a surface for a given first fundamental form and one component of its mapping [1]. Transposing this property to inextensional displacements gives the following expression of $\rho_{\alpha\beta}$:

Proposition 3.3. *For any infinitesimal bending* $\vec{u} = (u_1, u_2, u_3)$, *in Cartesian components, the linearized change of curvature tensor* $\rho_{\alpha\beta}(\vec{u})$ *depends only on the surface coefficients and one component of the displacement:*

$$\rho_{\alpha\beta}(\vec{u}) = \rho_{\alpha\beta}(u_3) = N^{\frac{1}{2}}\Big[u_3\big|_{\alpha\beta} + N(z_{,\lambda}u_{3,\mu}a^{\lambda\mu})z\big|_{\alpha\beta}\Big] \quad (15)$$

$$\text{with } N = \frac{1}{1 - z_{,\lambda}z_{,\mu}a^{\lambda\mu}} \text{ and } z\Big|_{\alpha\beta} = z_{,\alpha\beta} - \Gamma^{\nu}_{\alpha\beta}z_{,\nu}$$

The two expressions for $\rho_{\alpha\beta}$ in Eqs. (14) and (15) allow simplification and resolution of the asymptotic bending problem in Eq. (5) for specific cases.

3.2. Case of a hyperbolic paraboloid

Let S be a hyperbolic paraboloid and be defined by the mapping in Eq. (1) with

$$x = x_1, \quad y = x_2, \quad z = x_1x_2 \quad (16)$$

The covariant and contravariant basis and various coefficients of the surface are

$$\begin{aligned}
&\vec{a}_1 = (1,0,x_2) \qquad a_{11} = 1 + x_2^2 \qquad a^{11} = \frac{1+x_2^2}{a}\\
&b_{11} = 0 \qquad \Gamma^1_{12} = \frac{x_2}{a}\\
&\vec{a}_2 = (1,0,x_1) \qquad a_{12} = x_1x_2 \qquad a^{12} = \frac{x_1x_2}{a}\\
&b_{12} = \frac{1}{\sqrt{a}} \qquad \Gamma^2_{12} = \frac{x_1}{a}\\
&\vec{a}_3 = \frac{1}{\sqrt{a}}(-x_2,-x_1,1) \qquad a_{22} = 1 + x_1^2 \qquad a^{22} = \frac{1+x_2^2}{a}\\
&b_{22} = 0 \qquad \Gamma^{\beta}_{\alpha\alpha} = 0 \qquad (17)\\
&a = 1 + x_1^2 + x_2^2
\end{aligned}$$

and the elasticity coefficients for homogeneous and isotropic elastic shells are

$$A^{1111} = \frac{E(1+x_1^2)^2}{12(1-\nu^2)a^2} \qquad A^{1122} = \frac{E(x_1^2x_2^2 + \nu a)}{12(1-\nu^2)a^2}$$
$$A^{2222} = \frac{E(1+x_2^2)^2}{12(1-\nu^2)a^2} \quad (18)$$

The simplifications easily give the set of associated rotation fields and thus the set of inextensional displacements on a hyperbolic paraboloid [9]:

Proposition 3.4. *For any inextensional displacement* $\vec{u}$ *on the hyperbolic paraboloid S, there is a unique couple* $(\phi_{\vec{u}2}, \phi_{\vec{u}1}) \in \mathbb{L}^2_y \times \mathbb{L}^2_x$, *such that modulo a rigid displacement, we have*

$$\vec{u}(x_1,x_2) = R(\phi_{\vec{u}2}, \phi_{\vec{u}1}) = [x_2\Phi_{\vec{u}1}(x_1) - \Psi_{\vec{u}2}(x_2)]\,\vec{e}_1 + [\Psi_{\vec{u}1}(x_1) - x_1\Phi_{\vec{u}2}(x_2)]\,\vec{e}_2 - [\Phi_{\vec{u}1}(x_1) - [\Phi_{\vec{u}2}(x_2)]\,\vec{e}_3 \quad (19)$$

where the functions $\phi_{\vec{u}\alpha}$ and $\psi_{\vec{u}\alpha}$ *are defined with* $\phi_{\vec{u}\alpha}$ *by quadrature*:

$$\Phi_{\vec{u}\alpha}(x_1) = \int_0^x \phi_{\vec{u}\alpha}(z)\,(x_1 - z)dz \quad \text{and} \quad \Psi_{\vec{u}\alpha}(x_1) = \int_0^x \phi_{\vec{u}\alpha}(z)z\,(x_1 - z)dz \quad (20)$$

with $\rho_{11}(\vec{u}) = -\sqrt{a}\phi_{\vec{u}1}$, $\rho_{12}(\vec{u}) = 0$, *and* $\rho_{22}(\vec{u}) = \sqrt{a}\phi_{\vec{u}2}$

As an example, if we set $\Omega = [0, 1]^2$ and suppose the hyperbolic paraboloid is clamped along the boundary $x = 0$. This condition imposes that $\phi_{\vec{u}1} = 0$ in Eq. (19). The asymptotic bending problem simplifies then to a one-dimensional problem: with a unitary localized vertical loading on the point (0,1), we have

$$\begin{cases} \text{find } \phi_{x\vec{u}} \in \mathbb{L}^2_x([0,1]) \text{ such that} \\ \int_0^1 \alpha\phi_{\vec{u}}\phi_{\vec{v}}dx_1 = \Phi_{\vec{v}}(1) = \int_0^1 \phi_{\vec{v}}(x_1)(1-x_1)dx_1 \\ \forall \phi_{\vec{v}} \in L^2_x([0,1]) \end{cases} \quad (21)$$

where

$$\alpha(x_1) = \int_0^1 A^{1111}(x_1,x_2)a\sqrt{a}dy \quad (22)$$

This immediately gives the solution

$$\phi_{\vec{u}}(x_1) = \frac{1-x_1}{\alpha(x_1)} \quad \text{and} \quad u_3(1,0) = \int_0^1 \frac{(1-z)^2}{\alpha(z)}dz \quad (23)$$

With $E = 1$ and $\nu = 1/3$, we obtain $u_3\,(1, 0) = 3.5534$.

3.3. Case of a cylinder

Let us assume now that S is a cylinder. Let S be defined by the mapping in Eq. (1) with

$$x = x_1, \quad y = x_2, \quad z = \varphi(x_1). \quad (24)$$

The curve $(x_1, 0, \varphi(x_1))$ is a directrix and the curves (straight lines) at x_1 = constant are the generatrix. The covariant and contravariant basis and various coefficients of the surface are

$$\vec{a}_1 = (1,0,\varphi'(x_1)), \qquad a_{11} = a, \qquad b_{11} = \frac{\varphi''}{\sqrt{a}}$$
$$\Gamma^1_{11} = \frac{\varphi'\varphi''}{a}, \qquad a^{11} = \frac{1}{a},$$

$$\vec{a}_2 = (1,0,0), \quad a_{12} = 0, \quad b_{12} = 0,$$
$$\Gamma^2_{11} = 0, \quad a^{12} = 0, \tag{25}$$
$$\vec{a}_3 = \frac{1}{a}(-\varphi'(x_1),0,1), \quad a_{22} = 1, \quad b_{22} = 0,$$
$$\Gamma^{\alpha}_{\beta 2} = 0, \quad a^{22} = 1$$

with $a = 1 + (\varphi')^2$. The elasticity coefficients are

$$A^{1111} = \frac{E}{12a^2(1-\nu^2)}, \quad A^{1112} = 0, \quad A^{1212} = \frac{E}{a(1+\nu)} \tag{26}$$

In the following we shall suppose that φ'' doesn't vanish, in other words that S is not a plane. Then, it can be shown that the displacement u_3 is necessarily a polynomial in x_2 of degree 1:

$$u_3 = g(x_1) + x_2 h(x_1) \tag{27}$$

where g and h are arbitrary functions of x_1. Then,

$$\rho_{11} = N^{\frac{1}{2}}(u_{3,11} - \Gamma^1_{11}u_{3,1} + Nz_{,1}u_{3,1}a^{11}(z_{,11} - \Gamma^1_{11}z_{,1})),$$
$$\rho_{12} = N^{\frac{1}{2}}u_{3,12}, \quad \rho_{22} = 0 \tag{28}$$

Furthermore, as we have

$$Na^{11}\varphi'(\varphi'' - \Gamma^1_{11}\varphi') - \Gamma^1_{11} = 0, \tag{29}$$

we state:

Proposition 3.5. *Let $\vec{u}$ be an infinitesimal bending on the cylinder in Eq. (18), then $u_3 = g(x_1) + x_2h(x_1)$ and*

$$\rho_{11}(\vec{u}) = N^{\frac{1}{2}}(g'' + x_2h'') \quad \rho_{12}(\vec{u}) = N^{\frac{1}{2}}h' \quad \rho_{22}(\vec{u}) = 0 \tag{30}$$

Actually, in the case of inextensional displacement, it is possible to construct the components u_1 and u_2 from u_3 by quadratures from the equations of the bending system. Thus it is possible to express $L(\vec{u}) = L(u_3)$. The asymptotic bending problem in Eq. (5) can then be written as a variational problem of two functions of one variable:

$$\begin{cases} \text{Find } u_3 = g(x_1) + x_2h(x_1) \in \vec{G} \\ \int_S \frac{1}{12}\begin{bmatrix}\rho_{11}\\ 2\rho_{12}\end{bmatrix}^T \begin{bmatrix}A^{1111} & 0\\ 0 & A^{1212}\end{bmatrix}^T \begin{bmatrix}\rho^*_{11}\\ 2\rho^*_{12}\end{bmatrix} = L(u^*_3) \\ \forall u^*_3 = g^*(x_1) + x_2h^*(x_1) \in \vec{G} \end{cases} \tag{31}$$

It is then possible, for any cylindrical shells, to reduce the constrained problem to a one-dimensional differential problem.

As an example, consider $\Omega = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \times [-0.5, 0.5]$ and the cylinder defined with $z = \sqrt{1 - x_1^2}$, clamped on the boundary $x_1 = -\frac{\sqrt{2}}{2}$. It is easy to find that such configuration admits infinitesimal bending: all vertical displacements $u_3 = g(x_1) + x_2h(x_1)$ satisfying the boundary conditions $g(0) = g'(0) = h(0) = h'(0) = 0$ define admissible infinitesimal bendings. This geometry is also referred as a Scoderlis-Lo roof. The shell is submitted to a localized vertical force $\vec{F} = F\vec{e}_3$ on the point $p_3 = (\frac{\sqrt{2}}{2}, 0.5, \frac{\sqrt{2}}{2})$.

In this case, we don't have the exact analytical solution, but the reduced asymptotic bending problem in Eq. (5) is very simple, and it suffices to implement a standard one-dimensional finite element to solve it. We then obtain the value of the vertical component of the displacement on p_3 by numerical approximations, independent of the thickness. With $E = 1$, $\nu = 1/3$, $F = 1$, we have $u_3\,(p_3) = 2.1168107$.

4. Concluding remarks and membrane locking

Only two examples have been given here, but it is not difficult to construct more, with different shapes and loadings. These solutions can constitute tests for membrane locking: we compare with numerical results given by finite element schemes at given mesh for different thicknesses (going to 0).

Membrane locking in finite element computation of thin shells is the deterioration of the approximation for a fixed mesh when the thickness goes to 0 [10]. It is shown in [7] that any conformal finite element method is subject to membrane locking. Actually, the proof is also valid for some non-conformal methods such as DKT, but for non-conformal methods such as proposed in [11], including MITC [3], further analysis is needed. In the author's opinion, as convergence needs consistency, such methods should not be strictly locking-free, and the same applies to conformal methods, even though they seem to perform very well with respect to membrane locking [11].

In the particular case of localized loading, the deformation energy is given by the displacement at the loading point. It is easy then to compare the deformation energy: the asymptotic bending energy is necessarily smaller than the deformation energy for a given thickness. If present, the locking can then be detected in the specific case.

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