Inclusion properties for random relations under the hypotheses of stochastic independence and non-interactivity

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Abstract

This paper investigates whether random set inclusion is preserved by non-interactivity and by stochastic independence. Let (X_1, x_1), (X_2, x_2) be two random sets on U_1 and U_2, respectively, and let (Y_1, y_1), (Y_2, y_2) be two consonant inclusions of theirs. Let (Z_1, z_1) be the random relation on $U_1 \times U_2$ obtained from (X_1, x_1) and (X_2, x_2) under the hypothesis of stochastic independence, and let (Z_2, z_2) ((Z_3, z_3), resp.) be the random relation on $U_1 \times U_2$ obtained from (Y_1, y_1), (Y_2, y_2) under the hypothesis of non-interactivity (stochastic independence, resp.). We prove that these hypotheses do not imply that $(Z_1, z_1) \subseteq (Z_2, z_2)$, but imply that $(Z_1, z_1) \subseteq (Z_3, z_3)$.

Keywords: Random set; Relation; Inclusion; Non-interactivity; Stochastic independence

1. Introduction

Random set theory [1,2,3,4,5,6,7,8] provides a general paradigm for calculations with uncertain data. As illustrated in Fig. 1, let (Y_1, y_1), (Y_2, y_2) be two consonant inclusions of (X_1, x_1), (X_2, x_2), two random sets on U_1 and U_2, respectively. (Z_1, z_1) is the random relation for (X_1, x_1) and (X_2, x_2) using the hypothesis of stochastic independence, (Z_2, z_2) is the random relation for (Y_1, y_1) and (Y_2, y_2) using the hypothesis of non-interactivity and (Z_3, z_3) is the random relation for (Y_1, y_1) and (Y_2, y_2) using the hypothesis of stochastic independence. We prove that non-interactivity does not necessarily preserve inclusion and that stochastic independence preserves inclusion. The result has important consequences in the applications because consonant Cartesian random relations (which are equivalent to decomposable fuzzy relations) are much easier to handle from a computational viewpoint.

2. Inclusion of random set

Inclusion of random sets is defined as follows. Let (X_1, m_1) and (X_2, m_2) be two random sets: $(X_1, m_1) \subseteq (X_2, m_2)$ if and if only three conditions hold:

(i) $\forall A \in X_1, \exists B \in X_2, A \subseteq B$ (1)

(ii) $\forall B \in X_2, \exists A \in X_1, A \subseteq B$ (2)

(iii) there is a non-negative assignment matrix W with entries $W(A, B)$, $A \in X_1$, $B \in X_2$ such that:

$$\forall A \in X_1, m_1(A) = \sum_{B: A \subseteq B} W(A, B) \quad (3)$$

$$\forall B \in X_2, m_2(B) = \sum_{A: A \subseteq B} W(A, B) \quad (4)$$

where $W(A, B) = 0$ as soon as $A \not\subseteq B$.

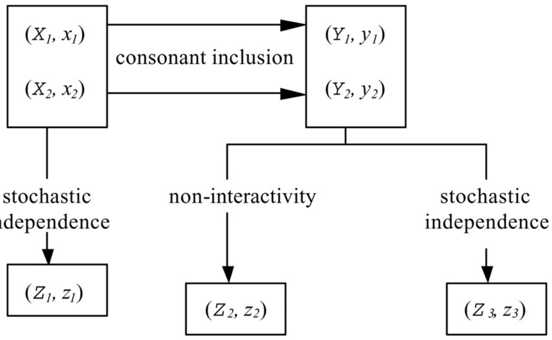

Fig. 1. Schematic of the random sets used in the paper.

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Dubois and Prade [6] showed that the inclusion of two random sets implies the inclusion of the [*Bel* (.), *Pl* (.)] interval:

$$(x_b m_1) \subseteq (x_2, m_2) \Rightarrow [Bel_1(A), Pl_1(A)] \subseteq [Bel_2(A), Pl_2(A)] \quad (5)$$

but the reverse is not necessarily true. Therefore:

$$[Bel_1(A), Pl_1(A)] \not\subset [Bel_2(A), Pl_2(A)] \Rightarrow (X_1, m_1) \not\subset (X_{2,} m_2) \quad (6)$$

3. Non-interactivity does not necessarily preserve inclusion

Let (X_1, x_1) and (X_2, x_2) be two marginal random sets respectively (Table 1). Following the procedure for inclusion of 1-D random sets given by Tonon [9], one obtains consonant random sets (Y_1, y_1) and (Y_2, y_2), which include (X_1, x_1) and (X_2, x_2), respectively (Table 2). Since each consonant random set (Y_i, y_i) is equivalent to a fuzzy set, F_i, focal elements A' and B' are also considered as α-cuts with α-levels μ_{F_1} (A') and μ_{F_2}, (B') for F_1 and F_2, respectively (Table 2).

Using the hypothesis of stochastic independence, random relation (Z_1, z_1) for (X_1, x_1) and (X_2, x_2) was calculated (Table 3). Using the hypothesis of non-interactivity, consonant random relation (Z_2, z_2) for (Y_1, y_1) and (Y_2, y_2) was also calculated and shown in Table 4.

Let us calculate the Belief–Plausibility intervals to determine whether (Z_1, z_1) ⊂ (Z_2, z_2). For $C_1 = D_1$, one obtains [Bel_{z_1} (C_1), Pl_{z_1} (C_1)] = [0.14,1] and [Bel_{z_2} (C_1), Pl_{z_2} (C_1)] = [0.19998,1]. Since Bel_{z_2} (C_1) > Bel_{z_1} (C_1), then [Bel_{z_1} (C_1), Pl_{z_1} (C_1)] $\not\subset$ [Bel_{z_2} (C_1), Pl_{z_2} (C_1)], one concludes that (Z_1, z_1) $\not\subset$ (Z_2, z_2)(Eq. 6)

4. Stochastic independence preserves inclusion

Let (Z_3, z_3) be the random relation obtained from marginals (Y_1, y_1), (Y_2, y_2) under the hypothesis of stochastic independence (Fig. 1). In this section we first show in general terms that (Z_1, z_1) ⊆ (Z_3, z_3), and then give a numerical example.

4.1. General case

One needs to show that the definition of inclusion given in Section 2 holds true.

(i) Recall that (X_i, x_i) ⊆ (Y_i, y_i) (i = 1, 2) and that the focal elements of (Z_1, z_1) ((Z_3, z_3), resp.) are Cartesian products of the focal elements of (X_i, x_i) ((Y_i, y_i), resp.) under the hypothesis of stochastic independence, i.e.

$$Z_1 = \{C_k | C_k = A_i \times B_j, A_i \in X_I B_j \in X_2\} \quad (7)$$

$$Z_3 = \{E_k | E_k = A'_i \times B'_j, A'_i \in Y_I B'_j \in Y_2\} \quad (8)$$

Let $C \in Z_1$ be such that $C = A \times B$, with $A \in X_1$, $B \in X_2$. Since (X_i, x_i) ⊆ (Y_i, y_i), ∃ $A' \in Y_1$, $B' \in Y_2$ such that $A \subseteq$

Table 1
Random sets (X_1, x_1), (X_2, x_2) with focal elements *A*, *B*

(X_1, x_1)		(X_2, x_2)	
A	$x_1(A)$	B	x_2 (B)
A_1 = [5, 8]	0.2	B_1 = [3, 7]	0.7
A_2 = [3, 7]	0.5	B_2 = [2, 5]	0.1
A_3 = [2, 4]	0.3	B_3 = [1, 8]	0.2

Table 3
Stochastically independent random Cartesian product (Z_1, z_1) with focal elements *C* obtained from (X_1, x_1) and (X_2, x_2)

$C = A \times B$	z_1 (C) = x_1 (A) x_2 (B)
$C_1 = A_1 \times B_1$ = [5,8] × [3,7]	0.14
$C_2 = A_1 \times B_2$ = [5,8] × [2,5]	0.02
$C_3 = A_1 \times B_3$ = [5,8] × [1,8]	0.04
$C_4 = A_2 \times B_1$ = [3,7] × [3,7]	0.35
$C_5 = A_2 \times B_2$ = [3,7] × [2,5]	0.05
$C_6 = A_2 \times B_3$ = [3,7] × [1,8]	0.1
$C_7 = A_3 \times B_1$ = [2,4] × [3,7]	0.21
$C_8 = A_3 \times B_2$ = [2,4] × [2,5]	0.03
$C_9 = A_3 \times B_3$ = [2,4] × [1,8]	0.06

Table 2
Random sets (Y_1, y_1) and (Y_2, y_2) with focal element A', B', respectively, and α-levels of their corresponding fuzzy sets F_1 and F_2, respectively

(Y_1, y_1)			(Y_2, y_2)		
A'	$y_1(A')$	$\mu_{F_1}(A')$	B'	$y_2(B')$	$\mu_{F_2}(B')$
A_1' = [5, 8]	0.199998	1	B_1' = [3, 7]	0.699998	1
A_2' = [3, 8]	0.5	0.800002	B_2' = [2, 7]	0.1	0.300002
A_3' = [2, 8]	0.300002	0.300002	B_3' = [1, 8]	0.200002	0.200002

Table 4
Non-interactive random Cartesian product (Z_2, z_2) with focal elements D_i obtained from (Y_1, y_1) and (Y_2, y_2) and its equivalent fuzzy sets

$D = A' \times B'$	$\mu_R(D)$	$z_2(D)$
$D_1 = A_1' \times B_1' = [5,8] \times [3,7]$	1	0.199998
$D_2 = A_2' \times B_1' = [3,8] \times [3,7]$	0.800002	0.5
$D_3 = (A_1' \times B_2') \cup (A_2' \times B_2') \cup (A_3' \times B_1') \cup (A_3' \times B_2') = [2,8] \times [2,7]$	0.300002	0.1
$D_4 = (A_1' \times B_3') \cup (A_2' \times B_3') \cup (A_3' \times B_3') = [2,8] \times [1,8]$	0.200002	0.200002

Table 5
Stochastically independent random Cartesian product (Z_3, z_3) with focal elements E_i obtained from (Y_1, y_1) and (Y_2, y_2)

$E = A' \times B'$	$z_3(E) = y_1(A')\, y_2(B')$
$E_1 = A_1' \times B_1' = [5,8] \times [3,7]$	0.139998200004
$E_2 = A_1' \times B_2' = [5,8] \times [2,7]$	0.0199998
$E_3 = A_1' \times B_3' = [5,8] \times [1,8]$	0.039999999996
$E_4 = A_2' \times B_1' = [3,8] \times [3,7]$	0.349999
$E_5 = A_2' \times B_2' = [3,8] \times [2,7]$	0.05
$E_6 = A_2' \times B_3' = [3,8] \times [1,8]$	0.100001
$E_7 = A_3' \times B_1' = [2,8] \times [3,7]$	0.210000799996
$E_8 = A_3' \times B_2' = [2,8] \times [2,7]$	0.0300002
$E_9 = A_3' \times B_3' = [2,8] \times [1,8]$	0.060001000004

A' and $B \subseteq B'$. Therefore, $C = A \times B \subseteq A' \times B' = E \in Z_3$.

(ii) Similar to (i).

(iii) First, the concept of convex sum of inclusion relationships [6] must be introduced. Consider a set Φ of 3-tuples $\{(A_i, B_i, W(A_i, B_i))\ i = 1,\ldots,k\}$, in which $W(A_i, B_i) > 0$ for $A_i \subseteq B_i$ and $\Sigma_i W(A_i, B_i) = 1$. The two random sets (X, m) and (X′, m') defined by $m(A) = \Sigma\{W(A_i, B_i)\ A_i = A\}$ and $m'(B) = \Sigma\{W(A_i, B_i)\ B_i = B\}$, are such that (X, m) $\subseteq$ (X′, m'). Hence this notion of inclusion corresponds to a convex sum of classical inclusion relationships. Note that the 3-tuples (A_i, B_i, $W(A_i, B_i)$) need not be distinct.

Second, the inclusions (X_i, x_i) $\subseteq$ (Y_i, y_i) (i = 1, 2) imply that there exist two matrices, say $W_1(A_i, A_i')$ and $W_2(B_j, B_j')$, respectively, which satisfy the definition of inclusion. Define the weight $W_3(A_i, B_j, A_i', B_j')$ as $W_1(A_i, A_i') \bullet W_2(B_j, B_j')$ and consider the set

$$\{((C_k = A_i \times B_j), (E_k = A_i' \times B_j'), W(A_i, B_j, A_i', B_j')) \\ |A_i \in X_1, B_j \in X_2, A_i' \in Y_1, B_j' \in Y_2\} \quad (9)$$

This set defines a convex sum of inclusion relationships because $W(A_i, B_j, A_i', B_j') > 0$ implies $W_1(A_i, A_i') > 0$ and $W_2(B_j, B_j') > 0$, which implies $A_i \subseteq A_i'$ and $B_j \subseteq B_j'$, which finally implies $A_i \times B_j \subseteq A_i' \times B_j'$. Now, one can conclude that

$$\sum_{A_i \times B_j = C_k} \left(\sum_{A_i': A_i \subseteq A_i'} W_1(A_i, A_i') \cdot \sum_{B_j': B_j \subseteq B_j'} W_2(B_i, B_j') \right) = \sum_{A_i \times B_j = C_k} x_1(A_i) \cdot x_2(B_j) = z_1(C_k) \quad (10)$$

Similarly,

$$\sum \{ W_l(A_i, A_i') \bullet W_2(B_j, B_j') | A_i' \times B_j' = E_k, A_i \in X_l, B_j \in X_2\} = Z_3(C_k') \quad (11)$$

From (i–iii) above, one concludes that (Z_1, z_1) $\subseteq$ (Z_3, z_3).

4.2. Numerical example

Requirements (i) and (ii) in the definition for random set inclusion can be easily checked using the first columns in Tables 3 and 5. A matrix W that satisfies requirement (iii) was found using a MATLAB optimization toolbox. Table 6 gives the zero and non-zero entries of matrix W. Table 7 is our exact $W(C_i, E_j)$ that meets the definition of inclusion of random sets (Z_1, z_1) $\subseteq$ (Z_3, z_3).

5. Conclusion and discussion

This paper shows that inclusion is not necessarily preserved by non-interactivity and that inclusion is preserved by stochastic independence.

From our result, one can find it is not possible to use the hypotheses of non-interactivity and stochastic independence interchangeably. If the inclusions are true, then the Belief–Plausibility intervals calculated with (Z_2, z_2) and (Z_3, z_3) include the Belief–Plausibility intervals calculated with (Z_1, z_1). If these bounds are interpreted as upper and lower probabilities, then probability bounds calculated with (Z_2, z_2) and (Z_3, z_3) include the probability bounds calculated with (Z_1, z_1). These bounds may be enough to make a decision on a system of interest and may grant substantial computational savings and robustness. For example, Tonon and Bernardini [10,11] used the hypothesis of non-interactivity

Table 6
Zero and non-zero entries in matrix $W(C_i,E_j)$

C_1	$W(C_1,E_1)$	$W(C_1,E_2)$	$W(C_1,E_3)$	$W(C_1,E_4)$	$W(C_1,E_5)$	$W(C_1,E_6)$	$W(C_1,E_7)$	$W(C_1,E_8)$	$W(C_1,E_9)$	0.14	$z_1(C_1)$
C_2	0	$W(C_2,E_2)$	$W(C_2,E_3)$	0	$W(C_2,E_5)$	$W(C_2,E_6)$	0	$W(C_2,E_8)$	$W(C_2,E_9)$	0.02	$z_1(C_2)$
C_3	0	0	$W(C_3,E_3)$	0	0	$W(C_3,E_6)$	0	0	$W(C_3,E_9)$	0.04	$z_1(C_3)$
C_4	0	0	0	$W(C_4,E_4)$	$W(C_4,E_5)$	$W(C_4,E_6)$	$W(C_4,E_7)$	$W(C_4,E_8)$	$W(C_4,E_9)$	0.35	$z_1(C_4)$
C_5	0	0	0	0	$W(C_5,E_5)$	$W(C_5,E_6)$	0	$W(C_5,E_8)$	$W(C_5,E_9)$	0.05	$z_1(C_5)$
C_6	0	0	0	0	0	$W(C_6,E_6)$	0	0	$W(C_6,E_9)$	0.1	$z_1(C_6)$
C_7	0	0	0	0	0	0	$W(C_7,E_7)$	$W(C_7,E_8)$	$W(C_7,E_9)$	0.21	$z_1(C_7)$
C_8	0	0	0	0	0	0	0	$W(C_8,E_8)$	$W(C_8,E_9)$	0.03	$z_1(C_8)$
C_9	0	0	0	0	0	0	0	0	$W(C_9,E_9)$	0.06	$z_1(C_9)$
	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9		
$z_3(E)$	0.139998200004	0.0199998	0.039999999996	0.349999	0.05	0.100001	0.210000799996	0.0300002	0.060001000004		

Table 7
Exact $W(C_i, E_j)$ that meets the definition of inclusion for random sets $(Z_1, z_1) \subseteq (Z_3, z_3)$

C_1	0.139998200004	0.00000002686243	0.00000009562753	0.00000012088093	0.00000033094530	0.00000031375302	0.00000014774336	0.00000073732100	0.00000002686243	0.14	$z_1(C_1)$
C_2	0	0.01999977313757	0.00000002686243	0	0.00000008173371	0.00000006454143	0	0.00000002686243	0.00000002686243	0.02	$z_1(C_2)$
C_3	0	0	0.03999987750604	0	0	0.00000009563153	0	0	0.00000002686243	0.04	$z_1(C_3)$
C_4	0	0	0	0.34999887911907	0.00000028918742	0.00000007199115	0.00000070597750	0.00000002686243	0.00000002686243	0.35	$z_1(C_4)$
C_5	0	0	0	0	0.04999929813357	0.00000048094530	0	0.00000002686243	0.00000019405870	0.05	$z_1(C_5)$
C_6	0	0	0	0	0	0.09999973313757	0	0	0.00000002686243	0.1	$z_1(C_6)$
C_7	0	0	0	0	0	0	0.20999946627514	0.00000002686243	0.00000002686243	0.21	$z_1(C_7)$
C_8	0	0	0	0	0	0	0	0.02999935522928	0.00000064477072	0.03	$z_1(C_8)$
C_9	0	0	0	0	0	0	0	0	0.06	0.06	$z_1(C_9)$
	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9		
$z_3(E)$	0.139998200004	0.0199998	0.039999999996	0.349999	0.05	0.100001	0.210000799996	0.0300002	0.060001000004		

in the formulation of a single multi-objective optimization of engineering systems. For more engineering applications of inclusion properties, please refer to the papers of Tonon et al. [9,12,13].

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