Superconvergence of *hp*-discontinuous Galerkin methods for convection-diffusion problems

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Abstract

We study some superconvergence properties of discontinuous Galerkin methods for convection-diffusion problems in one space dimension. We show that the nodal error converges with order $2p + 1$ if polynomials of degree p are used. The theoretical results are verified by numerical experiments.

Keywords: Superconvergence; Discontinuous Galerkin methods; Postprocessing

1. Introduction

In this note, we investigate the superconvergence properties of the *hp*-version of the DG method for convection-diffusion problems. We consider a steady-state model problem in one dimension. The method allows arbitrary meshes and arbitrary polynomial degrees. The main result of the paper is that for a particular choice of the numerical fluxes the error at the nodes converge with order $2p + 1$ if polynomials of degree p are used. This superconvergence property, with possibly slightly lower order, holds for general numerical fluxes as well. Analogous results were obtained by Douglas and Dupont [1] for the continuous version of the finite element method.

2. The problem and the main result

Consider the following model steady state convection-diffusion problem

$$-\varepsilon u'' + cu' = f \quad \text{in } \Omega = (0,1),$$
$$u(0) = u_D(0), \quad u(1) = u_D(1) \tag{1}$$

We assume that $c \geq 0$ and $\varepsilon > 0$. This can be rewritten as

$$q = \varepsilon u', \quad -(q - cu)' = f \ \text{in}\Omega,$$
$$u(0) = u_D(0), \quad u(1) = u_D(1) \tag{2}$$

Let $\mathcal{T} = \{I_j = (x_{j-1}, x_j), j = 1, \ldots, N\}$ be a triangulation of the computational domain $\Omega = (0, 1)$. The element width is defined as $h_j := x_j - x_{j-1}$ and we set $h = \max_{j=1}^{N} h_j$. We define $\Omega_h := \cup_{j=1,\ldots,N} I_j$, $(\varphi, \psi)_{I_j} := \int_{I_j} \varphi\psi$, $(\varphi, \psi)_{\Omega_h} := \sum_{I_j \subseteq \Omega_h} (\varphi, \psi)_{I_j}$, $\langle \varphi, \psi n \rangle_{\partial\Omega_h} := \sum_{j=1}^{N} \varphi\psi\big|_{x_{j-1}}^{x_j}$. For $k \geq 0$ we define the *broken* Sobolev seminorm $|\varphi|_{k,\Omega_h} := \left(\sum_{j=1}^{N} (\varphi^{(k)}, \varphi^{(k)})_{I_j}\right)^{\frac{1}{2}}$. The approximate DG solution $(u_h, q_h) \in V_h^p \times V_h^p$ is determined by requiring that

$$\begin{aligned} -(\varepsilon u_h, v')_{\Omega_h} &+ \langle \varepsilon \hat{u}_h, vn \rangle_{\partial\Omega_h} = (q_h, v) \\ (q_h - cu_h, w')_{\Omega_h} &- \left\langle \hat{q}_h - c\hat{\hat{u}}_h, wn \right\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h} \end{aligned} \tag{3}$$

for all $v, w \in V_h^p := \{v : \Omega_h \rightarrow \mathbb{R}: v|_{I_j} \in P^p(I_j), \forall I_j \in \mathcal{T}\}$, and $P^p(K)$ is the set of all polynomials on K of degree not exceeding p. The *numerical fluxes* are defined

$$(\hat{u}_h, \hat{\hat{u}}_h, \hat{q}_h)(x_j) = \begin{cases} (u_0, u_0, q_h(0^+)) & j = 0, \\ (u_h(x_j^-), u_h(x_j^-), q_h(x_j^+)) & 1 \leq j \leq N-1, \\ (u_1, u_h(1^-), q_h(1^-) - \alpha(u_h(1^-) - u_1)) & j = N. \end{cases} \tag{4}$$

where $\alpha = \epsilon \max\{1, p\}/h_N$.

Let x_i be an arbitrary but fixed interior node, define (φ_i, Φ_i) by the conditions $\Phi_i = \varphi'_i$, $(\epsilon\Phi'_i + c\varphi'_i) = 0$, $\varphi_i(0) = \varphi_i(1) = 0$ where φ_i and Φ_i are continuous on Ω except that Φ_i has a jump discontinuity of magnitunde $1/\epsilon$ at $x = x_i$.

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Theorem 2.1. *Consider the DG method defined by the weak formulation* (3) *and the numerical traces* (4). *Assume that* $(u, q) \in H^{s+2}(\Omega_h) \times H^{s+1}(\Omega_h)$ *for some* $s \geq 0$. *Then for any interior node* x_i *we have*

$$|(u - \hat{u}_h)(x_i)| \leq C|u|_{\sigma+2,\Omega_h} \max\left\{|\varphi_i|_{p+1,\Omega_h}, |\Phi_i|_{p+1,\Omega_h}\right\} \quad (5)$$

$$\left|(q - \hat{q}_h - c(u - \hat{\hat{u}}_h))(x_i)\right| \leq C|u|_{\sigma+2,\Omega_h} \max\left\{|\psi_i|_{p+1,\Omega_h}, |\Psi_i|_{p+1,\Omega_h}\right\}$$

for all $0 \leq \sigma \leq \min(s,p)$, *where*

$$C = C_s(4\varepsilon + \sqrt{c\varepsilon} + c)\frac{h^{\min(\sigma,p)+p+1}}{\max\{1,p\}^{\sigma+p+2}} \quad (6)$$

and C_s *is a constant depending solely on s.*
Proof. See [2]. □

This high order superconvergence at the nodes of the mesh can be exploited to obtain a better approximation which converges with order $2p + 1$ in the $L^2(\Omega)-$ norm. For some numerical examples of local (elemeny-by-element) postprocessing techniques we refer to [2]. It is worth noting that even though the original DG solution is discontinuous on Ω and converges with order $p + 1$ in the $L^2(\Omega)$-norm, the postprocessed solution is in $C^1(\Omega)$ and converges with order $2p + 1$.

We also note that this superconvergence phenomenon is also valid for more general numerical fluxes, for details see [2].

3. A numerical experiment

In this section we display some numerical results confirming Theorem 2.1. We consider (1) with $u(0) = u(1) = 0$, $c = 1$, $\epsilon = 0.1$, $f(x) = e^x$. In Table 1 we display convergence rates of the errors in the numerical fluxes. The first column shows the polynomial degree p we used to approximate the unknowns u_h and q_h. The second column displays the mesh number, where $mesh = i$ means we used a uniform mesh with 2^i elements. The numerical traces $\hat{u}_h$ and $\hat{q}_h - c\hat{\hat{u}}_h$ superconverge with order $2p + 1$, as was predicted by Theorem 2.1.

Table 1
Convergence rates for $c = 1$, $\epsilon = 0.1$ and $f(x) = e^x$

		$u - \hat{u}_h$		$q - \hat{q}_h - c(u - \hat{\hat{u}}_h)$	
Degree	Mesh	Error	Order	Error	Order
0	4	1.73E-01	0.77	3.23E-03	0.70
	5	9.52E-02	0.86	1.72E-03	0.91
	6	5.07E-02	0.91	8.81E-04	0.97
	7	2.62E-02	0.95	4.45E-04	0.99
1	4	2.02E-03	2.90	1.25E-06	3.04
	5	2.75E-04	2.88	1.56E-07	3.00
	6	3.56E-05	2.95	1.95E-08	3.00
	7	4.55E-06	2.97	2.44E-09	3.00
2	4	8.28E-06	4.92	8.98E-09	4.93
	5	2.76E-07	4.91	2.92E-10	4.94
	6	8.83E-09	4.97	9.36E-12	4.96
	7	2.80E-10	4.98	2.96E-13	4.98
3	4	1.69E-08	6.94	2.11E-11	6.94
	5	1.39E-10	6.92	1.70E-13	6.96
	6	1.11E-12	6.97	1.35E-15	6.97
	7	8.75E-15	6.98	1.07E-17	6.99
4	4	2.06E-11	8.95	2.61E-14	8.95
	5	4.23E-14	8.93	5.23E-17	8.97
	6	8.38E-17	8.98	1.04E-19	8.98
	7	1.65E-19	8.99	2.04E-22	8.99

4. Conclusion

In this paper, we investigated the superconvergence properties of the *hp* version of the DG method for a model convection-diffusion equation. We proved that, if the exact solution is sufficiently smooth, then the numerical fluxes converge with order $2p + 1$ when polynomials of degree p are used. Our numerical experiments verify this result and the optimality of it. Even though we considered a particular form of the numerical fluxes in this paper, the results are valid for more general DG methods. We also noted that the superconvergence phenomenon at discrete points can be exploited to obtained better approximations which converge globally with order $2p + 1$. Moreover, we can choose the postprocessed solution a smooth function in the overall computational domain, unlike the original DG solution which is only piecewise continuous.

References

[1] Douglas J, Dupont T. Galerkin approximations for the two point boundary problem using continuous piecewise polynomial spaces. Numer Math 1974;22:99–109.
[2] Celiker F, Cockburn B. Superconvergence and post-processing of *hp*-discontinuous galerkin methods for convection-diffusion problems (in preparation).