Numerical analysis of a quasistatic viscoplastic contact problem with friction and damage

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Abstract

In this paper, a frictional viscoplastic contact problem is studied. The damage, caused by excessive stress or strain is also included and it is modelled by a parabolic differential inclusion. The variational formulation for this problem is obtained and the existence of a unique solution is proved. Then, fully discrete approximations are introduced based on the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. Error estimates are derived and, under suitable regularity assumptions, the linear convergence of the algorithm is derived. Finally, a numerical test is provided.

Keywords: Viscoplasticity; Friction; Error estimates; Numerical simulations

1. Introduction

Frictional contact problems are very common in industry and everyday life. The contact of the tires with the road is just a simple example. For this reason, the engineering literature concerning this kind of problems is rather extensive (see, e.g., Laursen [1], Wriggers [2] and references therein).

The mechanical damage, caused by excessive stress or strain induced by loading forces, appears in many industrial problems involving frictional contact. Here, we used the damage model derived by Frémond and Nedjar [3] from thermodynamical principles. Recently, other damage models have been studied (see, e.g., Angelov [4], Liebe et al. [5] and Nedjar [6]).

In this work, a viscoplastic contact problem including friction and damage is considered. The friction is modelled using classical Tresca's law. The variational formulation leads to a coupled system of nonlinear variational inequalities. The existence of a unique weak solution is stated using fixed point arguments and classical results on variational inequalities. Then, a fully discrete scheme is introduced using the finite element method to approximate the spatial variable and an Euler scheme to discretize time derivatives. Using similar ideas to those applied in Chau et al. [7] and Chen et al. [8], a main error estimates result is proved from which, under suitable regularity assumptions, the linear convergence of the numerical scheme is deduced. Then, a numerical algorithm, based on the penalization of the frictional term, is implemented on an IBM RISC6000 computer, and some numerical results are performed.

2. The mathematical model and variational formulation

Let S_2 the space of second order symmetric tensors on $\mathbb{R}^2$, and we consider a viscoplastic body that occupies the domain $\Omega \subset \mathbb{R}^2$. We denote by $[O,T]$, $T > 0$, the time interval of interest. The outer surface $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_D, Γ_F and Γ_C. For a.e. $\mathbf{x} \in \Gamma$, we denote by $\boldsymbol{\nu}(\boldsymbol{x})$ and $\boldsymbol{\tau}(\boldsymbol{x})$ the unit normal and tangential vectors outward to Γ, respectively. A density of volume forces $\boldsymbol{f_B}$ acts in Ω and surface tractions of density $\boldsymbol{f_F}$ are given on Γ_F. The body is assumed to be clamped on Γ_D, and so the displacement field vanishes there. Finally, the body is assumed to be in contact with a foundation on the contact surface Γ_C.

Since the material is assumed viscoplastic, the following constitutive law is considered (see, e.g., Ionescu and Sofonea [9] and references therein):

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$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon(u), \zeta) \tag{1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor, $\boldsymbol{\sigma}$ is the respective stress tensor, and $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{G}}$ are the elastic tensor and the viscoplastic constitutive function, respectively, whose properties will be described below. Here, a dot above a variable represents its partial time derivative.

The damage of the material, denoted by ζ in Eq. (1), is defined in Ω and it measures the density of the microcracks: when $\zeta = 1$ the material is in its undamaged state, when $\zeta = 0$ the material is fully damaged and when $0 < \zeta < 1$ there is partial damage.

According to Frémond and Nedjar [3], the evolution of the damage is governed by the following parabolic nonlinear differential inclusion:

$$\dot{\zeta} - k\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \tag{2}$$

Here, $\triangle$ is the Laplacian, $\kappa > 0$ is the damage diffusion constant, ϕ is the damage source function and $\partial I_{[0,1]}$ denotes the subdifferential of the indicator function $\boldsymbol{I}_{[0,1]}$ of the interval [0,1].

We turn now to describe the contact boundary condition. We assume that the contact is bilateral and that it is associated to Tresca's law of friction. Therefore, we have

$$\left.\begin{array}{l} u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu} = 0, |\sigma_\tau| \le g \\ |\sigma_\tau| < g \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = g \Rightarrow \text{there exists } \lambda > 0 \text{ such that } \sigma_\tau = -\lambda\dot{u}_\tau \end{array}\right\} \text{on} \Gamma_C \times (0, T) \tag{3}$$

where g represents a friction bound and $u_\tau = \boldsymbol{u}\cdot\boldsymbol{\tau}$, $\sigma_\tau = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\tau}\cdot$

We denote by $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and ζ_0 the initial values of the displacement, stress and damage fields, respectively, and we assume that the inertia effects are negligible and that the process is quasistatic.

Let us define the following variational spaces:

$$V = \{\upsilon \in [H^1(\Omega)]^2; \boldsymbol{\upsilon} = \boldsymbol{0} \text{ on } \Gamma_D, \upsilon_\nu = \boldsymbol{\upsilon}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_C\}$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in [L^2(\Omega)]^{2\times 2}; \tau_{ij} = \tau_{ji}, i,j = 1,2\}$$
$$K = \{\xi \in H^1(\Omega); 0 \le \xi \le 1 \text{ a.e. in } \Omega\} \tag{4}$$

Let us assume that the elastic tensor $\mathcal{E} : \Omega \times S_2 \to S_2$ is a fourth-order symmetric definite positive tensor. Moreover, the viscoplastic function $\mathcal{G}$: $\Omega \times S_2 \times S_2 \times \mathbb{R} \to S_2$ and the damage source function $\o : \Omega \times S_2 \times S_2\, \mathbb{R} \to \mathbb{R}$ are assumed to be Lipschitz continuous, and the body forces and surface tractions have the regularity

$$\boldsymbol{f}_B \in W^{1,2}(0, T; [L^2(\Omega)]^2), \boldsymbol{f}_F \in W^{1,2}(0, T; [L^2(\Gamma_F)]^2) \tag{5}$$

Let $g : \Gamma_C \to [0, +\infty)$ be given such that $g\, \mathcal{E}\, L^\infty(\Gamma_C)$, $g \ge 0$ a.e. on Γ_C. Using Riesz's representation theorem, let $\boldsymbol{f}(t) \in V$ be given by the relation

$$(\boldsymbol{f}(t), \boldsymbol{\upsilon})_{\boldsymbol{V}} = (\boldsymbol{f}_B(t), \boldsymbol{\upsilon})_{[L^2(\Omega)]^2} + (\boldsymbol{f}_F(t), \boldsymbol{\upsilon})_{[\mathrm{L}^2(\Gamma_F)]^2}, \forall \boldsymbol{\upsilon} \in V \tag{6}$$

Let us define the following bilinear form $\alpha : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ by

$$a(\xi, \psi) = k\int_\Omega \nabla\xi\nabla\psi d\boldsymbol{x}, \forall \xi,\psi \in \mathcal{K} \tag{7}$$

and we denote by j: $V \to \mathbb{R}$ the functional ($\upsilon_\tau = \boldsymbol{\upsilon}\cdot\tau$)

$$j(\boldsymbol{\upsilon}) = \int_{\Gamma_C} g|\upsilon_\tau| da, \forall \boldsymbol{\upsilon} \in V \tag{8}$$

Finally, let the initial data $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and ζ_0 be chosen in such a way that $\boldsymbol{u}_0 \in V$, $\boldsymbol{\sigma}_0 \in Q$, $\zeta_0, \in \mathcal{K}$, and assume the following compatibility condition:

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{\upsilon}))_Q + j(\boldsymbol{\upsilon}) \ge (\boldsymbol{f}(0), \boldsymbol{\upsilon}), \forall \boldsymbol{\upsilon} \in V \tag{9}$$

Then, the following variational formulation is obtained:

Problem VP *Find a displacement field* $\boldsymbol{u}$: [0, T] $\to$ V, *a stress field* $\boldsymbol{\sigma}$: [0, T] $\to$ Q, *and a damage field* $\boldsymbol{\zeta}$: [0, T] $\to$ $\mathcal{K}$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$, $\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0$ *and for all* $\boldsymbol{\upsilon} \in V$, $\boldsymbol{\xi} \in \mathcal{K}$ *and a.e.* $t \in (0, T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\zeta}(t))$$
$$\boldsymbol{\sigma}(t), = \boldsymbol{\varepsilon}(\boldsymbol{\upsilon} - \dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{\upsilon}) - j(\dot{\boldsymbol{u}}(t)) \ge (\boldsymbol{f}(t), \boldsymbol{\upsilon} - (\dot{\boldsymbol{u}}(t))_V \tag{10}$$
$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{\upsilon} - \dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{\upsilon}) - j(\dot{\boldsymbol{u}}(t)) \ge (\boldsymbol{f}(t), \boldsymbol{\upsilon} - \dot{\boldsymbol{u}}(t))_V$$
$$(\phi(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_{L^2(\Omega)}$$

The following theorem states the existence of a unique solution to Problem VP:

Theorem 1 *Under the above assumptions, there exists a unique weak solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \zeta\}$ *to Problem VP with the following regularity*:

$$\boldsymbol{u} \in W^{1,2}(0, T; V), \quad \boldsymbol{\sigma} \in W^{1,2}(0, T; Q),$$
$$\zeta \in W^{1,2}(0, T; Y) \cap L^2(0, T; H^1(\Omega)) \tag{11}$$

The proof of Theorem 1 is done using fixed point arguments and classical results on variational inequalities (see Campo et al. [10] for details).

3. Numerical approximations

Let us assume that Ω is a polygonal domain and let $\mathcal{T}^h$ be a regular finite element triangulation of the domain Ω compatible with the boundary partition $\Gamma = \Gamma_D \cup$

$\Gamma_F \cup \Gamma_C$. We denote by θ^h the triangulation induced by $\mathcal{T}^h$ on Γ_C. Let us define the following variational spaces approximating V, $L^2(\Omega)$ and Q:

$$V^h = \{\boldsymbol{v}^h \in [C(\bar{\Omega})]^2;\ \boldsymbol{v}^h_{|T} \in [P_1(T)]^2 \quad \forall T \in T^h, \quad \boldsymbol{v}^h = \mathbf{0} \quad on \quad \Gamma_D$$
$$v^h_{v|c} = 0 \quad \forall C \in \theta^h\}$$
$$Y^h = \{\xi^h \in C(\bar{\Omega});\ \{\xi^h_{|T} \in P_1(T) \quad \forall T \in T^h\},$$
$$K^h = Y^h \cap K \tag{12}$$
$$Q^h = \{\boldsymbol{\tau}^h \in Q;\ \boldsymbol{\tau}^h_{|T} \in [P_0(T)]^{2\times 2} \quad \forall T \in T^h\}$$

Let $\mathcal{P}_Q{}^h$: $Q \rightarrow Q^h$ be the orthogonal projection operator defined through the relation

$$(\mathcal{P}_{Q^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\tau}, \boldsymbol{\tau}^h)_{Q,} \quad \forall \boldsymbol{\tau} \in Q,\ \boldsymbol{\tau}^h \in Q^h \tag{13}$$

In order to discretize the time derivatives, let $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition of the time interval $[0, T]$ and denote by k the time step, $k = T/N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we denote by $\delta w_n = (w_n - w^{n-1})/k$. In this section, no summation is considered over the repeated index n and, everywhere in the sequel, c will denote positive constants which are independent of the discretization parameters h and k.

Let $\boldsymbol{u}^h_0$, $\boldsymbol{\sigma}^h_0$ and ζ^h_0 be appropriate approximations of the initial conditions $\mathbf{u}_0$, $\boldsymbol{\sigma}_0$ and ζ_0, respectively. The fully discrete approximation is based on the forward Euler scheme and has the following form:

Problem VP^{hk} *Find a discrete displacement field*

$\boldsymbol{u}^{hk} = \{\boldsymbol{u}^{hk}_n\}_{n=0}^N \subset V^h$, *a discrete stress field*
$\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}^{hk}_n\}_{n=0}^N \subset Q^h$, *and a discrete damage field*
$\zeta^{hk} = \{\zeta^{hk}_n\}_{n=0}^N \subset \mathcal{K}^h$ *such that,* $\boldsymbol{u}^{hk}_0 = \boldsymbol{u}^h_0$, $\boldsymbol{\sigma}^{hk}_0 = \boldsymbol{\sigma}^h_0$, $\zeta^{hk}_0 = \zeta^h_0$, *and for* $n = 1, \cdots, N$, $\delta\boldsymbol{\sigma}^{hk}_n = \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^{hk}_n) +$

$$\mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}^{hk}_{n-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}), \zeta^{hk}_{n-1} \tag{14}$$
$$(\boldsymbol{\sigma}^{hk}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta\boldsymbol{u}^{hk}_n))_Q + j(\boldsymbol{v}^h) - j(\delta\boldsymbol{u}^{hk}_n) \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \delta\boldsymbol{u}^{hk}_n)_V, \quad \forall \boldsymbol{v}^h \in V^h$$
$$(\delta\zeta^{hk}_n, \xi^h - \zeta^{hk}_n)_Y + a(\zeta^{hk}_n, \xi^h - \zeta^{hk}_n) \geq (\phi(\boldsymbol{\sigma}^{hk}_{n-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}), \zeta^{hk}_{n-1}, \xi^h - \zeta^{hk}_n)_Y, \quad \forall \xi^h \in \mathcal{K}^h$$

Using classical results on variational inequalities, we obtain that Problem VP^{hk} admits a unique solution. Moreover, in Campo et al. [10] the following error estimates result was obtained:

Theorem 2 *Let the assumptions of Theorem 1 still hold. Let the initial conditions* $\boldsymbol{u}^h_0$, $\boldsymbol{\sigma}^h_0$ *and* ζ^h_0 *be defined by*

$$\boldsymbol{u}^h_0 = \Pi^h\boldsymbol{u}_{0,} \quad \boldsymbol{\sigma}^h_0 = \mathcal{P}_{Q^h}\boldsymbol{\sigma}_0, \quad \zeta^h_0 = \pi^h\zeta_0 \tag{15}$$

where $\pi^h : C(\bar{\Omega}) \rightarrow Y^h$ *is the standard finite element interpolation operator and* $\Pi^h = (\pi^h_i)^2_{i=1} : [C(\bar{\Omega})]^2 \rightarrow V^h$. *We also assume that*

$$\boldsymbol{u} \in C^1([0, T]; [H^2(\Omega)]^2), \quad \boldsymbol{\sigma} \in W^{1,\infty}(0, T; [H^1(\Omega)]^{2\times 2}), \quad \boldsymbol{\sigma\nu} \in C([0, T]; [L^2(\Gamma)]^2)$$
$$\zeta \in C[0, T]; H^2(\Omega)) \cap H^2(0, T; L^2(\Omega)), \quad \dot{\zeta} \in L^2(0, T; H^1(\Omega)) \tag{16}$$
$$\boldsymbol{u} \in H^2(0, T; V], \quad u_{\tau|C} \in H^1(0, T; H^2(C)) \quad \forall C \in \theta^h$$

Then, the linear convergence of the algorithm is obtained, i.e.

$$\max_{0\leq n\leq N}\left\{\left\|\boldsymbol{u}_n - \boldsymbol{u}^{hk}_n\right\|_V + \left\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}^{hk}_n\right\|_Q + \left\|\zeta_n - \zeta^{hk}_n\right\|_{L^2(\Omega)}\right\} \leq c(h + k) \tag{17}$$

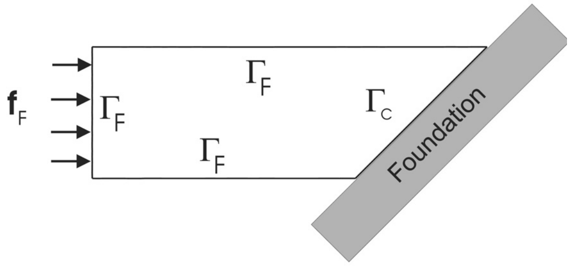

Fig. 1. An example with a large friction bound.

4. A numerical example with a large friction bound

In order to verify the performance of the numerical scheme described in the above section, the problem depicted in Figs. 1 has been considered.

The viscoplastic function G is a version of the constitutive function given by Maxwell:

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = -(1 - \zeta)\Phi(\boldsymbol{\sigma}) \tag{18}$$

where $\Phi: S_2 \rightarrow S_2$ is a truncation operator defined by

$$\forall \boldsymbol{\tau} = (\tau_{ij})^2_{i,j=1} \in S_2, \quad (\Phi(\boldsymbol{\tau}))_{ij} = \begin{cases} L & \text{if} \quad \tau_{ij} > L \\ \tau_{ij} & \text{if} \quad \tau_{ij} \in [-L, L] \\ -L & \text{if} \quad \tau_{ij} < -L \end{cases} \tag{19}$$

Here, $L = 1000$ is used.
The elastic tensor $\mathcal{E}$ is assumed homogeneous and satisfying the plane stress hypothesis with Young's modulus $E = 10^3$ N/m^2 and Poisson's ratio $r = 0.37$. The following data has been used:

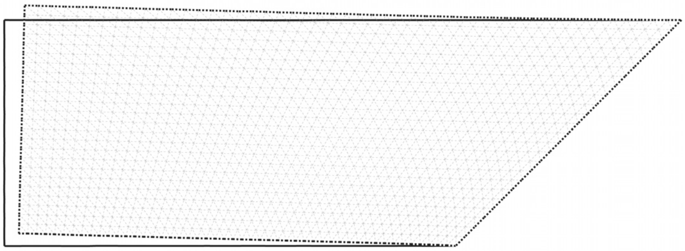

Fig. 2. Deformed mesh at final time and initial configuration.

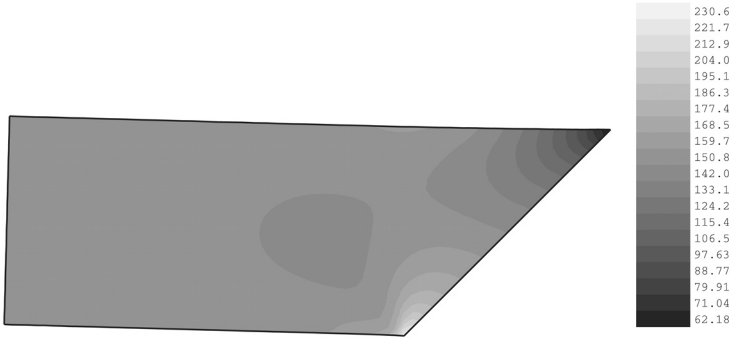

Fig. 3. Von Mises stress norm at $t = 1$ s.

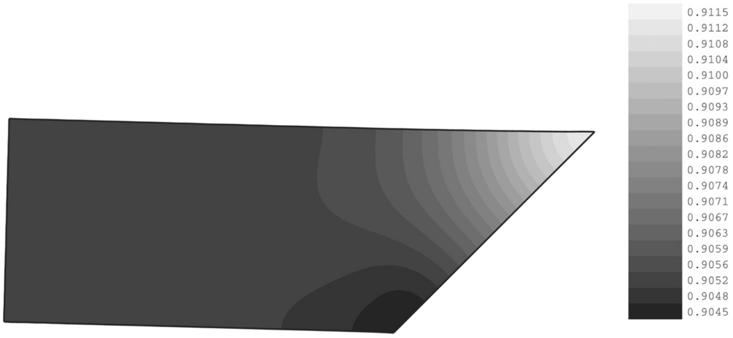

Fig. 4. Damage field at $t = 1$ s.

$$T = 1\,\mathrm{S}, \quad \boldsymbol{f}_N(\boldsymbol{x}, t) = (55, 0)e^t\,\mathrm{N/m^2}, \quad g = 1000\,\mathrm{N/m^2}, \quad \boldsymbol{u}_0 = \boldsymbol{0}\,\mathrm{m}, \quad \zeta_0 = 1 \tag{20}$$

The deformed mesh at final time and the initial configuration are plotted in Fig. 2, while the von Mises stress norm and the damage field at final time are shown in Figs. 3 and 4.

5. Conclusions

In this paper, a frictional viscoplastic contact problem has been studied. The effect due to the damage of the material was also considered. The variational formulation led to a system of nonlinear variational inequalities, and the existence of a unique weak solution was stated using fixed point arguments.

The aim was to provide a numerical analysis of the problem and to develop an efficient algorithm for solving it. A fully discrete scheme was introduced using the finite element method and the forward Euler scheme. Error estimates were derived and, as a consequence, under suitable regularity conditions, the linear convergence was obtained. The code was found to behave well and the numerical simulations seem accurate and interesting.

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