Stability results for the finite element approach to the immersed boundary method

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Abstract

The immersed boundary method in its original formulation, introduced by Peskin [1], exhibits a high degree of numerical stiffness. A finite element approach was proposed by Boffi et al. [2] and further developed in Boffi et al. [3,4] to solve some of the difficulties arising from the original formulation. We present a stability analysis for the two dimensional case with some numerical tests that show how to choose the numerical parameters in order to obtain optimal stability.

Keywords: Immersed boundary method; Finite element method; Stability analysis

1. Introduction

The original numerical approach to the IBM is based on finite differences for the spatial discretisation, leading to the construction of a suitable approximation of the Dirac delta function, which is used to take into account the interaction equations (see [1,5]).

Our approach to the discretization of the IBM is completely based upon the finite element method [2–4]. Our aim is to deal with the delta function, which is related to the forces exerted by the immersed structure on the fluid and viceversa, in a variational way, so that there is no need to construct its regularization, but its effect is taken into account by its action on the test functions.

We will recall the IBM variational formulation and present some stability results for a two dimensional case, together with some numerical tests.

2. Setting of the problem

Let Ω be the two- or three-dimensional domain containing the fluid and the flexible or elastic structure.

The original formulation of the immersed boundary method introduces a 'non-standard' source term in the Navier–Stokes equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla p=\mathbf{F} \quad \text{in } \Omega\times]0,T[\qquad (1)$$
$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega\times]0,T[$$

which requires the use of a Dirac delta function:

$$\mathrm{F}(\mathrm{x},t)=\int_\Omega \mathbf{f}(\mathbf{q},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{q},t))\mathbf{dq}, \quad \text{in } \Omega\times]0,T[\qquad (2)$$

Our starting point will be the variational formulation of the IBM, as given in Boffi et al. [2–4]. We consider the model problem of a viscous incompressible fluid in a simple two- or three-dimensional domain Ω containing an immersed massless elastic boundary in the form of a curve or a surface.

Problem 1 *Given* $\mathbf{f}\in L^2(D\times]0,T[)$, $\mathbf{u}_0 \in H^1_0(\Omega)^d$ and $\mathbf{X}_0 : D\to\Omega$, *for all* $t\in]0,T[)$, *find* $(\mathbf{u}(t),p(t)) \in H^1_0(\Omega)^d\times L^2_0(\Omega)$ *and* $\mathbf{X} : D\times]0,T[\to\Omega$, *such that* $\rho\big(\frac{d}{dt}(\mathbf{u}(t),\mathbf{v})+(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})\big)+\mu(\nabla\mathbf{u}(t),\nabla\mathbf{v})$

$$-(\nabla\cdot\mathbf{v},p(t))=<\mathbf{F}(t),\mathbf{v}> \qquad \forall v\in H^1_0(\Omega)^d \qquad (3)$$
$$(\nabla\cdot\mathbf{u}(t),\ q)=0 \qquad \forall q\in L^2_0(\Omega) \qquad (4)$$
$$<\mathbf{F}(t),\mathbf{v}>=\int_D \mathbf{f}(\mathbf{s},t)\mathbf{v}(\mathbf{X}(\mathbf{s},t))\,\mathbf{ds} \qquad \forall v\in H^1_0(\Omega)^d \qquad (5)$$
$$\frac{\partial\mathbf{X}}{\partial t}(\mathbf{s},t)=\mathbf{u}(\mathbf{X}(\mathbf{s},t),t) \qquad \forall s\in D \qquad (6)$$

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$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \qquad \forall x \in \Omega \qquad (7)$$

$$\mathbf{X}(\mathbf{s},0) = \mathbf{X}_0(\mathbf{s}) \qquad \forall s \in D \qquad (8)$$

The density force formulation (2) is substituted by (5), where the Dirac delta function is no longer needed.

Equation (7) represents the initial conditions relative to the variational formulation of Navier-Stokes equations (3)–(4), where we imposed homogeneous Dirichlet boundary conditions; other boundary conditions could also be used. Equation (5) expresses the force exerted by the structure on the fluid in terms of the force density $\mathbf{f}(\mathbf{q}, t)$. Equation (8) is the initial condition for Eq. (6), which drives the motion of the immersed structure.

Thanks to the generality of Eq. (2), the immersed boundary method can produce robust numerical schemes in order to simulate complex fluid–structure interaction systems. We will consider the simple model problem of a viscous incompressible fluid in a two-dimensional square domain Ω containing an immersed massless boundary in the form of a curve (see, e.g. [6,7]). To be more precise, for all $t \in [0, T]$, let Γ_t be a simple closed elastic curve, the configuration of which is given in a parametric form, $\mathbf{X}(s, t)$, $0 \leq s \leq L$, $\mathbf{X}(0, t) = \mathbf{X}(L, t)$. The force exerted by the element of boundary ds on the fluid is $\mathbf{f}(s, t)ds$. When dealing with linear elasticity and homogeneous materials, the force density $\mathbf{f}$ can be written as follows:

$$\mathbf{f}(s,t) = \kappa \frac{\partial^2 \mathbf{X}}{\partial s^2}(s,t) \qquad (9)$$

3. Stability results for the discrete problem

The following stability estimate holds true:

Lemma 1 *For $t \in]0, T[$, let $\mathbf{u}(t) \in H_0^1(\Omega)^2, p(t) \in L_0^2(\Omega)$ and $\mathbf{X}(t) \in (H^1(D))^2$ be a solution of Problem 1, then it holds that*:

$$\frac{\rho}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_0^2 + \mu\|\nabla\mathbf{u}(t)\|_0^2 + \frac{\kappa}{2}\frac{d}{dt}\left\|\frac{\partial \mathbf{X}}{\partial s}(t)\right\|_0^2 = 0 \qquad (10)$$

where $\|\cdot\|_0$ stands for the norm in $L^2(\Omega)$.

Equation (10) represents the energy equilibrium of the system and shows how the elastic energy of the immersed boundary is transferred to the fluid.

Let T_h be a subdivision of Ω into triangles or rectangles. We denote by h_x the biggest diameter of the elements of T_h. We then consider two finite dimensional subspaces $\mathbf{V}_h \subseteq H_0^1(\Omega)^2$ and $Q_h \subseteq L_0^2(\Omega)$. It is well known that the pair of spaces $\mathbf{V}_h$ and Q_h need to satisfy the inf-sup condition in order to have existence, uniqueness and stability of the discrete solution of the Navier-Stokes problem (3)–(4) [8,9].

Next, let s_i, $i = 0, \ldots, m$ with $s_0 = 0$ and $s_m = L$, be $m + 1$ distinct points of the interval $[0, L]$. We set $h_s = \max_{0 \leq i \leq m} | s_i - s_{i-1} |$. Let $\mathbf{S}_h$ be the finite element space of piecewise linear vectors defined on $[0, L]$ as follows:

$$\mathbf{S}_h = \{\mathbf{Y} \in C^0([0,L];\Omega : \mathbf{Y}|_{[s_{i-1},s_i]} \in \mathcal{P}^1([s_{i-1},s_i])^2,\ i = 1,\ldots,m, \quad \mathbf{Y}(s_0) = \mathbf{Y}(s_m)\} \qquad (11)$$

where $\mathcal{P}^1(I)$ stands for the space of affine polynomials on the interval I. For an element $\mathbf{Y} \in \mathbf{S}_h$ we shall use also the following notation $\mathbf{Y}_i = \mathbf{Y}(s_i)$ for $i = 0, \ldots, m$.

The first step, in order to introduce the discrete counterpart of Problem 1, is the computation of Eq. (5) for all $\mathbf{X}_h \in \mathbf{S}_h$ and for all $\mathbf{v} \in \mathbf{V}_h$. It can be shown that in case of linear elasticity and with a piecewise linear discretisation of the immersed boundary $\mathbf{X}_h$, Eq. (5) can be reformulated as follows:

$$< \mathbf{F}_h(t),\mathbf{v} >= \sum_{i=0}^{m-1} \kappa\left(\frac{\partial \mathbf{X}_{hi+1}}{\partial s}(t) - \frac{\partial \mathbf{X}_{hi}}{\partial s}(t)\right)\mathbf{v}(\mathbf{X}_{hi}(t)) \qquad (12)$$

Notice that the right-hand side of Eq. (12) is meaningful, since $\mathbf{v}$ is continuous as it is required for the elements in $\mathbf{V}_h$. The stability estimate introduced in Eq. (10) holds also for a density force expressed as in Eq. (12). For a derivation of Eq. (12) refer to Boffi et al. [4].

To solve numerically Problem 1 we neglected the non linear terms in Navier-Stokes equations (3), discretized the domain in a uniform grid of squares, used the stable Q2/P1 finite element pair for the space discretization and backward Euler method for advancing in time. In this simplified setting the fully discrete problem reads:

Problem 2 *Given $\mathbf{f} \in L^2(]0, L[\times]0, T[)$, $\mathbf{u}_{0h} \in \mathbf{V}_h$ and $\mathbf{X}_{0h} \in \mathbf{S}_h$, set $\mathbf{u}_h^0 = \mathbf{u}_{0h}$ and $\mathbf{X}_h^0 = \mathbf{X}_{0h}$, then for $n = 0, 1, \ldots, N-1$*

Step 1. *Compute the source term*

$$< \mathbf{F}_h^{n+1},\mathbf{v} >= \sum_{i=0}^{m-1} \kappa\left(\frac{\partial \mathbf{X}_{hi+1}^n}{\partial s} - \frac{\partial \mathbf{X}_{hi}^n}{\partial s}\right)\mathbf{v}(\mathbf{X}_{hi}^n) \qquad \forall \mathbf{v} \in \mathbf{V}_h \qquad (13)$$

Step 2. *Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times Q_h$, such that*

$$\rho\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t},\mathbf{v}\right) + \mu(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}) - (\nabla\cdot\mathbf{v},p_h^{n+1}) =< \mathbf{F}_h^{n+1},\mathbf{v} > \qquad \forall v \in V_h$$
$$(\nabla\cdot\mathbf{u}_h^{n+1},q) = 0 \qquad \forall q \in Q_h \qquad (14)$$

Step 3. *Find $\mathbf{X}_h^{n+1} \in \mathbf{S}_h$, such that*

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^{n}}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n}) \qquad \forall i = 1, \ldots, m \qquad (15)$$

In our numerical experiments we observed that the immersed boundary mesh size h_s should not be chosen arbitrarily small with respect to the fluid mesh size h_x and to the time step Δt.

It appears that in order to reduce the area loss of the boundary (which is one of the measures used to evaluate the accuracy of the method) it is advisable to have at least two nodes of the boundary per element. On the other hand we observed that the method accuracy does not improve when raising the number of boundary nodes to more than four per domain element.

In this work we show the sensitivity of the method to the semi-explicit time stepping technique used to move the boundary in Step 3 of the discrete formulation.

In particular we observed that when h_s is chosen too small with respect to h_x, it is necessary to decrease also the time step Δt to ensure the stability of the method. The conditions that h_x, h_s and Δt have to satisfy are shown in the following lemmas.

Lemma 2 *Let* Δt *be such that for all* $n = 0, \ldots, N-1$

$$\mu - \frac{C^2\kappa}{2}\frac{\Delta t}{h_s h_x}\sum_{i=0}^{m-1}\left|\mathbf{X}_{hi}^{n} - \mathbf{X}_{hi-1}^{n}\right| \geq K_0 > 0, \qquad (16)$$

then for all $n = 1, \ldots, N$

$$\frac{\rho}{2}\|u_h^n\|^2 + \Delta t\sum_{k=1}^{n} K_0\|\nabla \mathbf{u}_h^k\|^2 + \frac{\kappa}{2}h_s\sum_{i=0}^{m-1}\frac{\left|\mathbf{X}_{hi}^{n} - \mathbf{X}_{hi-1}^{n}\right|^2}{h_s^2}$$
$$\leq \frac{\rho}{2}\|\mathbf{u}_{0h}\|^2 + \frac{\kappa}{2}h_s\sum_{i=0}^{m-1}\frac{\left|\mathbf{X}_{hi}^{0} - \mathbf{X}_{hi-1}^{0}\right|^2}{h_s^2} \qquad (17)$$

Figure 1 shows the evolution of a boundary when the correct ratio between h_s, h_x and Δt is used, while Fig. 2 is an example of the kind of instability we obtain when the immersed boundary is too refined with respect to the fluid domain and the time step, where N stands for the immersed boundary node density and M for the fluid domain element density.

Several computations have been done, showing the correctness of Lemma 2. We report a significant snapshot of a stability test in Fig. 3. In a forthcoming paper we will demonstrate these results presenting a full stability analysis, together with some extensions to the results presented in Boffi et al. [2–4].

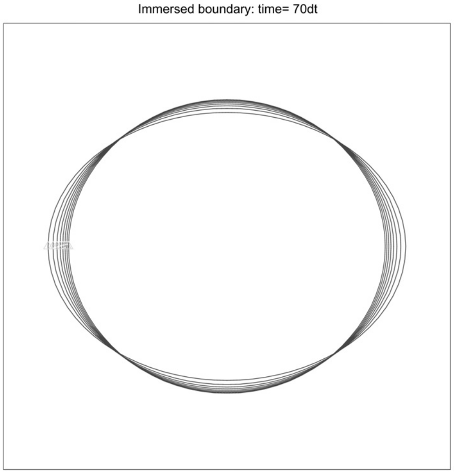

Fig. 1. Correct evolution of a two-dimensional boundary with N = 16, M = 35 and Δt = 0.04.

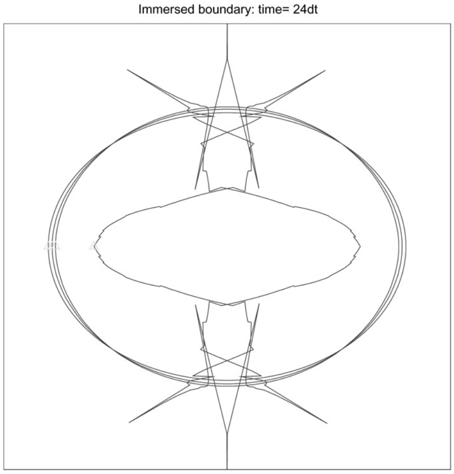

Fig. 2. Unstable evolution of a two-dimensional boundary with N = 16, M = 545 and Δt = 0.04.

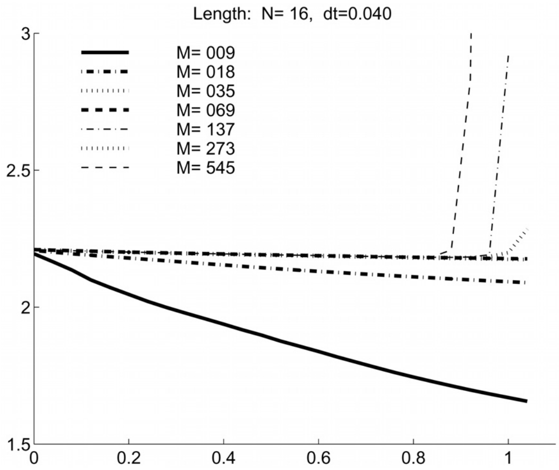

Fig. 3. Snapshot of the instability in the case of a two-dimensional immersed boundary.

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