An analytical formulation for the prediction of buckling and post-buckling of composite panels and shells

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Abstract

This paper presents an analytical formulation for buckling and post-buckling prediction of composite panels and shells. Based on Donnel-Von Karman and Kirchhoff hypotheses, the formulation takes into account for an equilibrate and symmetric composite laminate model. After introducing the Airy functions, the equilibrium and compatibility equations are solved using the Galerkin method. The buckling and post-buckling solutions are obtained for flat panels, loaded by compression or shear, and for cylindrical shells, loaded by compression. The effect of the initial geometric imperfections is also considered. The analytical solutions are validated by non-linear finite element analyses.

Keywords: Analytical formulation; Buckling; Post-buckling field; Composite panels and shells; Geometric imperfections; Finite element analyses

1. Introduction

The use of composite materials in aerospace structural components is extremely attractive due to their considerable strength-to-weight ratio, but unfortunately their use is not yet very extensive and their design is often overly conservative. For example, panels and cylindrical shells in composite materials are not allowed to work in the post-buckling field and also the predicted buckling loads of shell structures are reduced by a 'knock-down' factor that takes into account the influence of imperfections and of unintended deviation from nominal values.

To obtain widely applicable design criteria, the methodology of integrating a reasonable number of experiments with a complementary computational activity seems to be the only viable approach. Indeed, nowadays the improvements in the computational analysis tools make possible more sophisticated analytical and numerical models for the non-linear response, allowing also modeling the initial geometric imperfections. In any case, the analytical and numerical models need to be validated with test results [1,2], before they can be used with confidence.

Finite element models have been developed that allow the post-buckling behavior of panels and shells in composite materials [3] to be investigated efficiently, but they are slow computational methods that require a long time to calibrate the model and long CPU time. So analytical methods that can give with good accuracy the buckling loads and an idea of the post-buckling field are of great interest from the industrial point of view, especially during the preliminary design phase.

This paper presents an analytical formulation that allows determining not only the buckling loads but also the post-buckling field of flat panels subject to compression or shear and of cylindrical shells subject to axial compression.

2. Equilibrium and compatibility equations for composite panels and shells

The equilibrium and compatibility equations are written in tensor notation [4] for generical panels of anysotropic materials, starting from an analytical theory developed in the 1960s [5]. The equations are written referring to the midplane x, y of the panels, having the thickness h much smaller than the other two dimensions and small curvature. The following approximations are taken into account:

- approximation of Donnel-Von Karman, for which

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the displacements are considered infinitesimal except the displacement $w(x, y)$ normal to the midplane;
- hypothesis of Kirchhoff, for which the normals to the midplane are maintained.

The buckling configuration is looked for starting from an initial pre-loaded configuration, where the pre-stresses are represented by the tensors ${}_Ep^{ik}$. The initial imperfections are taken into account through the tensor ${}_ow_{/ik}$ and the panel curvature through the tensor b_{ik}. The Airy function $\chi = \chi(x, y)$ is introduced so that the equation unknowns are the Airy function $\chi(x, y)$ and the normal displacement $w(x, y)$.

The classical laminate theory [6] is then introduced, considering a symmetric laminate:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{21} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{61} & D_{62} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{1}$$

where $\{\varepsilon_o\}$ represents the midplane strains, $\{k\}$ the midplane curvatures, $\{N\}$ and $\{M\}$ the stress resultants (in the form of forces per unit length and moments per unit length).

For simplicity only symmetric and equilibrate laminates are considered, so that the matrix $[A]$ that connects $\{N\}$ and $\{\varepsilon_o\}$ becomes:

$$\{N\} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} \tag{2}$$

while its inverse becomes:

$$\left[\tilde{A}\right] = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & 0 \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \tag{3}$$

Another approximation in the laminate is considered: the laminates that are symmetric and equilibrate present also the terms $D_{16} = D_{26} = 0$.

So the equilibrium equation for flat laminate panels gives:

$$\begin{aligned}
&\left({}_Ep^{xx}w_{/xx} + {}_Ep^{yy}w_{/yy} + 2{}_Ep^{xy}w_{/xy}\right)+\\
&\quad\left({}_Ep^{xx}{}_ow_{/xx} + {}_Ep^{yy}{}_ow_{/yy} + 2{}_Ep^{xy}{}_ow_{/xy}\right)+\\
&\quad\left(\chi_{/yy}w_{/xx} - \chi_{/yx}w_{/xy} - \chi_{/xy}w_{/yx} + \chi_{/xx}w_{/yy}\right)+\\
&\quad\left(\chi_{/yyo}w_{/xx} - \chi_{/yxo}w_{/xy} - \chi_{/xyo}w_{/yx} + \chi_{/xxo}w_{/yy}\right)-\\
&\quad D_{11}w_{/xxxx} - D_{22}w_{/yyyy} - 2(D_{12} + 2D_{66})w_{/xxyy} = 0
\end{aligned} \tag{4}$$

while the equilibrium equations for cylindrical shells, where $b_{ik} = b_{xx} = -\frac{1}{R}$, gives:

$$\begin{aligned}
&-{}_Ep^{xx}\frac{1}{R} + \left({}_Ep^{xx}w_{/xx} + {}_Ep^{yy}w_{/yy} + 2{}_Ep^{xy}w_{/xy}\right)+\\
&\quad\left({}_Ep^{xx}{}_ow_{/xx} + {}_Ep^{yy}{}_ow_{/yy} + 2{}_Ep^{xy}{}_ow_{/xy}\right) + \ -\frac{1}{R}\chi_{/yy}+\\
&\quad\left(\chi_{/yy}w_{/xx} - \chi_{/yx}w_{/xy} - \chi_{/xy}w_{/yx} + \chi_{/xx}w_{/yy}\right)+\\
&\quad\left(\chi_{/yyo}w_{/xx} - \chi_{/yxo}w_{/xy} - \chi_{/xyo}w_{/yx} + \chi_{/xxo}w_{/yy}\right)-\\
&\quad D_{11}w_{/xxxx} - D_{22}w_{/yyyy} - 2(D_{12} + 2D_{66})w_{/xxyy} = 0
\end{aligned} \tag{5}$$

The compatibility equation for flat laminate panels gives:

$$\begin{aligned}
&\left[\tilde{A}_{11}\chi_{/yyyy} + \tilde{A}_{22}\chi_{/xxxx} + 2\left(\tilde{A}_{12}\chi_{/xxyy} + 0.5\tilde{A}_{33}\chi_{/xxyy}\right)\right]+\\
&\quad\left(w_{/xx}w_{/yy} - w_{/xy}w_{/yx}\right) - \frac{1}{R}w_{/yy}+\\
&\quad+\left(w_{/yyo}w_{/xx} - w_{/yxo}w_{/xy} - w_{/xyo}w_{/yx} + w_{/xxo}w_{/yy}\right) = 0
\end{aligned} \tag{6}$$

while the compatibility equations for cylindrical shells gives:

$$\begin{aligned}
&\left[\tilde{A}_{11}\chi_{/yyyy} + \tilde{A}_{22}\chi_{/xxxx} + 2\left(\tilde{A}_{12}\chi_{/xxyy} + 0.5\tilde{A}_{33}\chi_{/xxyy}\right)\right]+\\
&\quad\left(w_{/xx}w_{/yy} - w_{/xy}w_{/yx}\right)+\\
&\quad+\left(w_{/yyo}w_{/xx} - w_{/yxo}w_{/xy} - w_{/xyo}w_{/yx} + w_{/xxo}w_{/yy}\right) = 0
\end{aligned} \tag{7}$$

3. Flat panels in composite materials subject to compression or shear

Simply supported flat panels are considered subjected to two different load conditions: compression and shear.

First, the expression of linear buckling load is obtained, together with the buckling mode. Because of the complexity of the closed-solution of the equilibrium and compatibility equations, the method of Galerkin is applied, representing the unknowns by the infinite double trigonometrical series:

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\,\mathrm{sen}\left(\frac{\pi m x}{a}\right)\mathrm{sen}\left(\frac{\pi n y}{b}\right) \tag{8}$$

$$\chi(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mu_{mn}\,\mathrm{sen}\left(\frac{\pi m x}{a}\right)\mathrm{sen}\left(\frac{\pi n y}{b}\right) \tag{9}$$

where m is the number of half-waves in the x direction, n the number of the half-waves in the y direction, a and b are the length of panel sides in the x and y direction, respectively, and the panel is loaded in compression along the x direction.

In the case of compression, the expression of the linear buckling stress is given by:

$$\sigma_{buckling} = \frac{2}{h}\left(\frac{\pi}{b}\right)^2\left(\sqrt{D_{11}D_{22}} + D\right) \quad (10)$$

where $D = (D_{12} + 2D_{66})$.
The buckling mode under compression is given by $n_c = 1$ and $m_c = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{0.25}$

In the case of shear, the expression of the linear buckling stress is given by:

$$\tau = \frac{1}{2ht}\left[\left(\frac{\pi}{l}\right)^2\left(D_{11} + t^4D_{22} + 2t^2D\right) + \left(\frac{\pi}{b}\right)^2\left(6t^2D_{22} + 2D\right) + \left(\frac{\pi l}{b^2}\right)^2 D_{22}\right] \quad (11)$$

for the values of l and t that minimize τ, where l is the length of the single buckling wave, while t is the tangent of the inclination angle β of the wave. Equation 11 is minimized case by case using MATLAB.

Then, for the panels loaded in compression, the post-buckling field is studied considering also the effect of the initial geometric imperfections, that have the same shape of the buckling mode:

$$_{o}w(x,y) = d\,sen\left(\frac{\pi m_c x}{a}\right)sen\left(\frac{\pi n_c y}{b}\right) \quad (12)$$

In this case, Eqs. (4) and (6) are considered taking into account also the non-linear terms, applying the Galerkin method and obtaining the value of the compression stress σ versus q:

$$\sigma = \frac{1}{(q+d)}\frac{1}{h}\left\{\left[\sigma_{buckling}h + \left(\frac{1024}{9}\hat{C}\right)d^2\right]q + \left[\left(\frac{512}{3}\hat{C}\right)d\right]q^2 + \left[\frac{512}{9}\hat{C}\right]q^3\right\} \quad (13)$$

where:

$$\hat{C} = \frac{1}{\hat{A}}\left(\frac{\pi m_c}{a}\right)^2\left(\frac{\pi n_c}{b}\right)^4\frac{1}{\pi^4}\frac{1}{m_c^2 n_c^2} \quad (14)$$

$$\hat{A} = \frac{1}{\left(A_{11}A_{22} - A_{12}^2\right)}\left[\left(\frac{\pi m_c}{a}\right)^4 A_{11} + \left(\frac{\pi n_c}{b}\right)^4 A_{22} + \left(\frac{A_{11}A_{22} - A_{12}^2}{A_{33}} - 2A_{12}\right)\left(\frac{\pi m_c}{a}\right)^2\left(\frac{\pi n_c}{b}\right)^2\right] \quad (15)$$

and the shortening u_x versus q:

$$u_x = \left[\frac{A_{22}}{\left(A_{11}A_{22} - A_{12}^2\right)}h\right]\sigma - \left\{\left[2\frac{\hat{B}}{\hat{A}} - \frac{1}{4}\left(\frac{\pi m_c}{a}\right)^2\right]d\right\}q + \left[\frac{1}{8}\left(\frac{\pi m_c}{a}\right)^2 - \frac{\hat{B}}{\hat{A}}\right]q^2 \quad (16)$$

where:

$$\hat{B} = \frac{16}{3}\frac{1}{\pi^4 m_c^2 n_c^2}\left(\frac{\pi m_c}{a}\right)^2\left(\frac{\pi n_c}{b}\right)^2\frac{1}{\left(A_{11}A_{22} - A_{12}^2\right)}\left[4A_{22}\left(\frac{\pi n_c}{b}\right)^2 - 4A_{12}\left(\frac{\pi m_c}{a}\right)^2\right] \quad (17)$$

4. Flat panels in composite materials subject to compression or shear: application and comparison to finite element analyses

An application of the formulation is given for flat panel with $a = b = 300$ mm in carbon fiber and epoxy matrix. Two panels with four layers and lay-up sequences equal to $[0°/90°]_S$ and $[0°/45°/-45°/0°]$ are analyzed. The panel thickness is equal to 1.32 mm.

The σ versus u_x curves, for different values of the adimensional parameter f of the initial geometric imperfection, defined as $f = d/h$, is given in Fig. 1, for the $[0°/90°]_S$ panel.

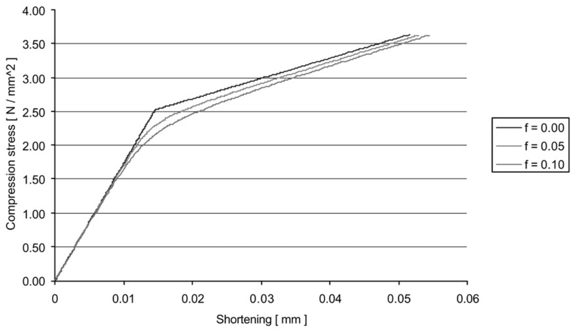

Fig. 1. Panel $[0°/90°]_S$: stress-shortening curve for different values of initial imperfections.

The analytical results are then compared with those obtained from finite element analyses using the commercial code ABAQUS. The buckling loads and the buckling mode are obtained through eigenvalue analyses, while the stress-shortening curves are obtaining by means of non-linear Riks method [3]. The differences between the buckling loads obtained through the analytical formulation and the finite element analyses are contained in 2% in the case of compression load and in 4% in the case of shear load. An example of the stress-shortening curves is reported in Fig. 2. It represents the comparison in the case of a panel $[0°/90°]_S$ with an initial imperfection amplitude $f = 0.1$.

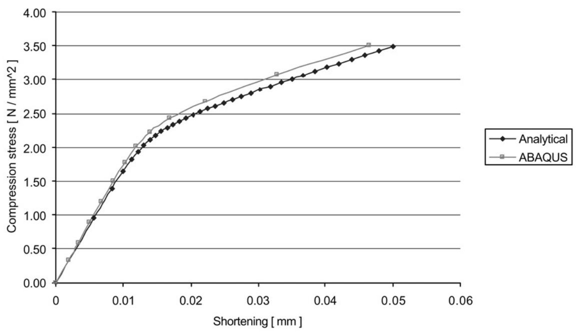

Fig. 2. Panel $[0°/90°]_S$: Comparison between analytical and numerical stress-shortening curve for an initial imperfection amplitude $f = 0.1$.

5. Cylindrical shells in composite materials subjected to compression

Cylindrical shells subjected to axial compression are analyzed.

First, the expression of linear buckling load is obtained, together with the buckling mode, applying the *Galerkin method* where only the first term of the series for the unknowns w and χ is considered.

The expression of the linear buckling stress is given by:

$$\sigma = \frac{1}{h}\left(\frac{R}{n}\right)^2\left[\left(\frac{m}{R}\right)^4 D_{11} + \left(\frac{n}{R}\right)^4 D_{22} + 2\left(\frac{m}{R}\right)^2\left(\frac{n}{R}\right)^2 D\right] + \\ + \frac{1}{\left[\left(\frac{m}{R}\right)^4 \tilde{A}_{22} + \left(\frac{n}{R}\right)^4 \tilde{A}_{11} + 2\left(\frac{m}{R}\right)^2\left(\frac{n}{R}\right)^2 \tilde{A}\right]}\frac{1}{h}\left(\frac{n}{R}\right)^2\frac{1}{R^2} \quad (18)$$

for the values of m and n that minimize σ, where m is the number of waves in the circumferential direction and n the number of waves in the axial direction. The minimization is performed case by case using MATLAB.

Then, the post-buckling field is studied considering the non-linear terms and also the effect of the initial geometric imperfections. Two different analytical formulations are developed: the first one for the cylindrical shells that present an axialsymmetric buckling mode and the second one for those that present a diamond buckling mode.

For example, in the case of diamond buckling mode the expression of the compression stress σ and of the shortening u_x versus q, are given respectively by:

$$\sigma = \frac{1}{(q+2d)}\frac{1}{h}\left\{\left[\sigma_{buckling} h + \hat{a}\frac{1}{\pi^2 R}\frac{64}{3}d + \hat{a}\left(\frac{N_c}{R}\right)^2\frac{1}{\pi^4}\frac{1024}{9}d^2\right]q + \left[\hat{a}\frac{1}{\pi^2}\frac{16}{R} + \hat{a}\left(\frac{N_c}{R}\right)^2\frac{1}{\pi^4}\frac{512}{3}d\right]q^2 + \left[\hat{a}\left(\frac{N_c}{R}\right)^2\frac{1}{\pi^4}\frac{512}{9}\right]q^3\right\} \quad (19)$$

$$u_y = \left[\frac{A_{11}}{\left(A_{11}A_{22} - A_{12}^2\right)}h\right]\sigma + \\ -\left\{\hat{a}\hat{b}\left(\frac{R}{N_c}\right)^2\frac{1}{\pi^2}\frac{1}{R} + \left[\hat{a}\hat{b}\frac{1}{\pi^4}\frac{32}{3} - \frac{1}{2}\left(\frac{N_c}{R}\right)^2\right]d\right\}q + \\ -\left[\hat{a}\hat{b}\frac{1}{\pi^4}\frac{16}{3} - \frac{1}{8}\left(\frac{N_c}{R}\right)^2\right]q^2 \quad (20)$$

where N_C is the number of diamond wave in a length equal to the circumference in correspondence of the buckling load.

6. Cylindrical shells in composite materials subjected to compression: application and comparison to finite element analyses

An application of the formulation is given for cylindrical shells with radius $R = 350$ mm and the same laminates of the already considered flat panels.

The σ versus u_y curves, for different values of the adimensional parameter f of the initial geometric imperfection, are given in Fig. 3, for the $[0°/45°/-45°/0°]$ shell.

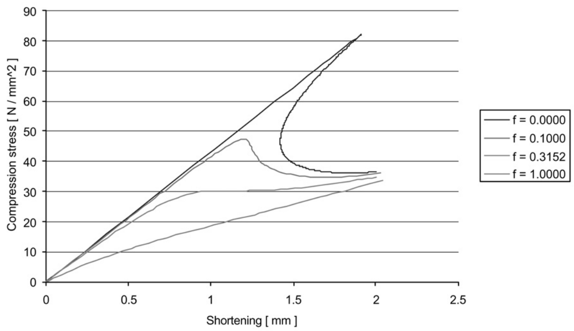

Fig. 3. Shell $[0°/45°/-45°/0°]$: stress-shortening curve for different values of initial imperfections.

The analytical results are then compared with those obtained from ABAQUS analyses. The differences between the buckling loads obtained through the analytical formulation and the finite element analyses are lower than 4% when no initial geometric imperfections are considered. They are a little bit higher in the case of geometric imperfections, as shown in Fig. 4, where the

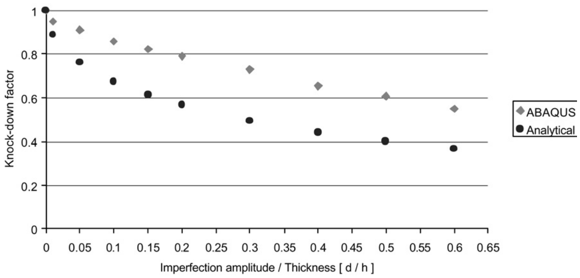

Fig. 4. Shell $[0°/90°]_S$: Comparison between analytical and numerical knock-down factors for diamond geometric imperfections.

knock-down factors are reported in the case of a $[0°/90°]_S$ shell with diamond initial imperfections.

7. Conclusions

The analytical formulation here proposed appears able to correctly predict both buckling loads and post-buckling field for flat panels and cylindrical composite shells. The comparisons with the finite element results show a good agreement in the cases of two structural configurations here analyzed. The obtained results appear very promising for the extension of the proposed formulation to generic composite laminates. As typical aerospace structures such as wing and fuselage are mainly based on panels and shells, the proposed analytical formulation can be considered a very fast and efficient tool. Indeed, especially during the preliminary design phase, it is important to rapidly investigate different structural configurations, and detailed but expensive non-linear finite element analyses can not be applied, due to their requests in terms of computation and mesh generation time.

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