Fundamental and applicative challenges in the modeling and computations of shells

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Abstract

We discuss the effectiveness and reliability of shell finite element procedures in relation to the asymptotic behaviors of shell structures when varying the thickness as a parameter. This is crucial in order to understand the complexity and diversity of shell physical behavior, and the difficulties to be confronted in the finite element analysis. In addition, we present some results and challenges directly related to applicative concerns and numerical practice, in particular as regards general shell elements and 3D-shell elements.

Keywords: Shells; Asymptotic behavior; Numerical locking; Mixed formulations, General shell elements

1. Introduction

The purpose in this paper is to survey some fundamental concepts that are very important in order to understand the physical behavior of shell structures and the numerical phenomena associated with their finite element discretizations. An essential underlying idea is that a deep synergy between physical and mathematical understanding is necessary in order to effectively analyze shell structures.

In Section 2, we give an outline of the asymptotic behaviors of shells when varying the thickness of the structure while keeping the mid-surface and boundary conditions unchanged. In Section 3, we discuss the difficulties in obtaining uniformly optimal finite element procedures for shells, in particular due to the variety of asymptotic behaviors that may arise. We then present, in Section 4, some specific problems related to the finite element analysis of shells in engineering applications. Finally, we give some concluding remarks in Section 5.

2. Asymptotic behaviors of shell models: an outline

In this section we present a ‘roadmap’ to the asymptotic behaviors of shells. For more details, see [1] and references therein.

2.1 Motivations and setting

As is well-known in engineering practice, shell structures may produce dramatically different responses – especially when the shell thickness is rather small compared to other characteristic dimensions – depending on their geometries and boundary conditions, in particular. The key to understanding these phenomena is to analyze their *asymptotic behaviors*, namely by considering a sequence of problems indexed by the thickness parameter that we vary while keeping the midsurface geometry and boundary conditions fixed. We write this sequence of problems in the following variational form:

$$\varepsilon^3 A_{\mathrm{b}}(U^\varepsilon, V) + \varepsilon A_{\mathrm{m}}(U^\varepsilon, V) = F^\varepsilon(V), \ \forall V \in \mathcal{V} \quad (1)$$

The meanings of the symbols appearing in this formulation are:

- ε: a dimensionless thickness parameter, namely, the actual thickness t – assumed to be constant for simplicity here, without loss of generality [1] – divided by an overall dimension of the structure;
- U^ε: the unknown solution, namely the displacement of the midsurface for a membrane-bending (m-b) shell model, or this displacement *and* the rotation of the normal fiber for a shear-membrane-bending (s-m-b) model;
- $\mathcal{V}$: the Sobolev space in which we seek the solution (we recall that the definition of this space takes into account the essential boundary conditions);

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- V: a test function;
- A_b: a scaled representation of the bending energy;
- A_m: a scaled representation of the membrane energy for an m-b shell model, or of the membrane energy *and* of the shear energy for an s-m-b model;
- $F^{\varepsilon}(V)$: the external virtual work associated with V.

We emphasize that *the bilinear forms A_b and A_m do not depend on the thickness parameter ε*. In addition, we also introduced ϵ as a superscript in the right-hand side of the formulation because it is in general impossible to obtain a well-posed asymptotic behavior while keeping the loading constant over the whole sequence of problems. More specifically, what we are after in the asymptotic analysis is a scaling of the right-hand side in the form

$$F^{\varepsilon}(V) = \varepsilon^{\rho} G(V) \tag{2}$$

where G is an element of $\mathcal{V}'$ *independent of* ε and ρ is a real number, for which the scaled external work $G(U^{\varepsilon})$ converges to a *finite and non-zero* limit when ε tends to zero.

The behavior of U^{ε} when ε tends to zero crucially depends on the contents of a specific subspace of $\mathcal{V}$, namely,

$$\mathcal{V}_0 = \{V \in \mathcal{V} \,|A_m(V, V) = 0\} \tag{3}$$

which we call the subspace of *pure bending displacements* – since only the bending part of the energy is non-zero for these displacements – and also the subspace of *inextensional displacements* because the key in satisfying the constraint A_m (V,V) is that the midsurface membrane strains vanish, namely that the midsurface displacements be inextensional. In the following discussion we distinguish two cases according to whether or not this subspace is reduced to the zero element. This strongly depends on the geometry of the midsurface considered and on the boundary conditions enforced, see [1,2] for detailed examples of both cases.

2.2. Bending-dominated shells

Here we assume that the subspace $\mathcal{V}_0$ contains non-zero elements. Then it can be shown that we obtain an admissible asymptotic behavior by considering the scaling

$$F^{\varepsilon}(V) = \varepsilon^{3} G(V) \tag{4}$$

In this case, U^{ϵ} converges – for the norm of V – to U^0, the solution of
Find $U^0 \in \mathcal{V}_0$ such that

$$A_b\,(U^0,\ V) = G(V), \quad \forall V \in \mathcal{V}_0 \tag{5}$$

and we say that the shell structure is bending-dominated because all the deformation energy goes to the bending part in the asymptotic limit [1,3,4].

2.3. Membrane-dominated shells

By contrast, we assume in this section that the inextensional subspace $\mathcal{V}_0$ is reduced to the zero element. This is quite frequent in practice because the inextensional constraints correspond to three scalar differential equations to be satisfied by the three components of displacements, hence these constraints are 'strong'.

Then we can define an inner product – and the corresponding norm – using the bilinear form A_m, as follows:

$$\| V \|_m = \sqrt{A_m\,(V,\ V)} \tag{6}$$

We call this norm the *membrane energy norm* and we define $\mathcal{V}_m$ as the space obtained by completion of $\mathcal{V}$ for this norm (essentially, $\mathcal{V}_m$ is the space of all displacements with bounded membrane energy).

In this case we can show (see [4,5]) that an admissible behavior is obtained for the scaling

$$F^{\varepsilon}(V) = \times \varepsilon G(V) \tag{7}$$

namely, U^{ε} converges – for the membrane energy norm – to U^m, the solution of
Find $U^m \in \mathcal{V}_m$ such that

$$A_m(U^m,\ V) = G(V), \quad \forall V \in \mathcal{V}_m \tag{8}$$

This convergence result holds provided that G is in the dual space of $\mathcal{V}_m$, viz.

$$|G(V)| \le C\sqrt{A_m\,(V,\ V)}, \quad \forall V \in \mathcal{V} \tag{9}$$

which gives a restriction on the loading. When this condition is satisfied we say that the shell structure is *membrane-dominated* since the membrane energy becomes increasingly dominant when the thickness tends to zero. We point out that in this case the stiffness of the structure is 'in ε' – as opposed to ε^3 in the bending-dominated case – hence we have a drastically different (much stiffer) response. When Eq. (9) does not hold we say that we have an ill-posed membrane problem, see [1,6,7] for examples thereof.

3. Reliability of finite element methods: the 'asymptotic dilemma'

Of course, when performing finite element analyses of shell structures we would like the finite solutions to accurately reflect the diversity of the above-described behavior. More precisely, since we only discretize the problem over the midsurface (i.e. not across the thickness), we expect an accuracy that would only depend on criteria prevailing in 2D analysis, namely with relative error bounds of the type

$$\frac{\| U^{\varepsilon} - U_h^{\varepsilon} \|_*}{\| U^{\varepsilon} \|_*} \leq Ch^p \tag{10}$$

where U_h^{ε} represents the finite element solution for a given thickness value ε, and where the bounding constant C and the order of convergence p *should not depend on* ε. This means that we expect *uniform convergence* of the finite element solution with respect to the thickness parameter. In the above equation we denote the norm with a '*' symbol to indicate that the norm for which uniform convergence is expected may differ according to the specific asymptotic case considered (typically this norm will be the norm for which the asymptotic behavior of the exact solution is well-posed), and the order p should then be the optimal order of convergence for this norm (namely the order of convergence of interpolation errors). Such a uniform estimate is very important to ensure the *reliability* of the finite element procedure considered.

However, it was soon recognized in the development of structural analysis procedures that standard finite element techniques – such as displacement-based shell finite elements – fail to display such uniformly converging behaviors in general, and that instead finite element approximations tend to dramatically deteriorate when the thickness of the structure decreases. In fact, when pursuing the above reliability objectives one faces a dilemma which we now explain.

When considering a bending-dominated structure, the numerical difficulty to deal with is *numerical locking*, since the asymptotic behavior then corresponds to a penalized formulation such as for nearly-incompressible elasticity. In order to treat locking, one is led to resorting to mixed formulations which can be summarized as

$$A_{\mathrm{b}}(U_h^{\varepsilon}, V) + \frac{1}{\varepsilon^2} A_{\mathrm{m}}^h (U_h^{\varepsilon}, V) = G(V), \ \forall V \in \mathcal{V}_h \tag{11}$$

where A_{m}^h denotes a perturbed form of A_{m} that corresponds to a relaxed form of the constraints prevailing in the pure bending subspace at the discrete level. However, uniform error bounds for such a formulation rely on satisfying a crucial *discrete inf-sup condition* which has never been established for any shell finite element scheme so far, except under very restrictive assumptions on the geometry [8]. In fact, the only procedure for which uniform estimates have been obtained is the *mixed stabilized* formulation [9] which – indeed – is designed to dispense with the inf-sup condition. Nevertheless, some detailed numerical assessments using carefully designed test problems allow us to identify specific shell procedures that show little sensitivity of the convergence behavior with respect to the thickness parameter [1,10]. This holds in particular for the MITC4 shell element [11].

By contrast, when considering a membrane-dominated structure displacement-based finite element schemes can be shown to provide uniformly optimal estimates [4]. However, since the asymptotic behavior can seldom be determined *a priori* in complex applications, one is led to using a mixed formulation in all practical situations in order to circumvent locking when applicable, which means that the discrete formulation solved for a membrane-dominated structure is

$$A_{\mathrm{m}}^h (U_h^{\varepsilon}, V) + \varepsilon^2 A_{\mathrm{b}} (U_h^{\varepsilon}, V) = G(V), \ \forall V \in \mathcal{V}_h \tag{12}$$

Here we can see that we perturb by the mixed procedure the membrane term, namely, the essential part of the energy in the asymptotic behavior. Therefore, in order to ensure convergence we need to enforce adequate consistency properties on this perturbation. This is a considerable difficulty, essentially because the 'logic' of unlocking is deeply foreign to that of membrane consistency and – as an example – the above-mentioned mixed stabilized formulation which has been mathematically substantiated in the bending-dominated regime does not enjoy adequate membrane consistency features. This is why we refer to an 'asymptotic dilemma' as regards shell finite elements.

Of course, as already mentioned, carefully designed numerical assessment can be resorted to in order to progress towards solving this dilemma. This should be particularly useful in identifying effective *triangular elements* – note that all elements tested in [1,10] are quadrilateral – which are much needed in applications where complex structures and unstructured meshes are often considered, see [12] for some preliminary results in this direction.

4. Problems arising from numerical practice in applications

Given the considerable impact of shell finite element procedures in engineering applications, it is essential to be able to consider – and sometimes also formulate – shell finite element procedures that are designed to fulfill certain applicative specifications, motivations or criteria, whereas these procedures may be very difficult to analyze from a mathematical point of view. This is the case – in particular – with general shell elements and with 3D-shell elements which we now discuss.

4.1. General shell elements and their underlying model

General shell elements – henceforth referred to by the acronym GSEs – are most widespread in today's engineering practice (MITC elements, among others, belong to this family [13]). However, they have long been considered as a mystery, in particular for applied mathematicians. These elements – indeed – rely on the discretization of a *3D* variational formulation, not of a shell mathematical model. In fact this justifies their name since this methodology allows – in principle at least – to obtain shell finite elements for an arbitrary 3D mechanical formulation (in particular as regards the constitutive law) without going through the stage of shell mathematical modeling. In a way we could thus say that GSEs perform 'computer-aided shell modeling', which is also true for facet shell elements, another family commonly encountered in practice [1,14].

In [15] (see also [1]) we have shown the existence of a mathematical shell model underlying GSEs by obtaining error estimates – converging with respect to the mesh size – between the approximate GSE solution and the exact solution of this well-identified mathematical model. Moreover, we have shown that this underlying model compares well with classical shell models (see in particular [2]) in the sense of asymptotic consistency. Namely, when the thickness tends to zero the solutions of the underlying model converge to the same limits – and under similar assumptions – to those of classical models. This also establishes asymptotic consistency of the underlying model with 3D elasticity [2]. Therefore these results provide both a mathematical justification of GSEs and a bridge between them and discretizations of classical shell models which have been largely studied in the mathematical literature, see in particular [16] and references therein.

4.2. 3D-shell elements and models

The '3D-shell elements' that we have proposed and analyzed in [17] belong to the GSE family. Their name refers to the fact that both their geometrical definition and their shape functions are those of standard isoparametric 3D elements, namely, with only *displacement* degrees of freedom (no rotations) ascribed to nodes also located on the upper and lower surfaces of the shell (i.e. not only on the midsurface). More specifically, their formulation is based on a prismatic geometry (e.g. hexahedra) and on corresponding shape functions taken *quadratic* with respect to the through-the-thickness variable.

These shell elements are designed to address the following applicative motivations:

- Since they are similar to 3D elements as regards their shape functions and (isoparametric) geometrical definition, 3D-shell elements are most easily coupled to other 3D elements through their external facets. This can be most conveniently taken advantage of when modeling the coupling of a shell with a solid (such as reinforcement layers, e.g. in a tyre, sandwich structures, piezoelectric patches, and so on) or in fluid–structure interaction.
- The underlying (quadratic) kinematics of these elements is 'richer' than in usual GSEs (for which Reissner-Mindlin kinematics is considered). This allows us to capture more accurately some physical phenomena associated with large deformations where the transverse deformation is crucial, such as for metal forming.
- It can be shown that 3D-shell elements do not require (unlike usual GSEs) a plane stress assumption, hence the 3D variational formulation can be used 'as is'.

We have extended to 3D-shell elements the mathematical results already mentioned for GSEs, namely, that there exists a well-posed underlying mathematical shell model which is asymptotically consistent with classical shell models (hence also with 3D formulations).

Furthermore, we have shown that the same sources of locking already identified in other shell elements are present in 3D-shell elements, hence they can be handled by the same strategies (in particular by using MITC procedures) [1]. In addition, a new source of locking arises in these elements in association with the transverse deformation energy. However we have been able to prove [18] that this difficulty – which we called 'pinching locking' – can be effectively treated by using a simple mixed interpolation strategy already proposed for other high-order shell elements [19]. It is important to note that unlocking treatments applied on 3D-shell elements characterize the only difference between these elements and actual 3D elements. Nevertheless, this is a very important difference because it allows to use these elements as *shell elements*, namely for *thin structures*.

Incidentally, we point out that a new difficulty arises in 3D-shell elements when considering incompressible – or nearly-incompressible – formulations, since the singularity corresponding to incompressibility is retained in the (unmodified) 3D energy. This led us to investigating – and partly substantiating – the concept of an *incompressible shell* in [20].

5. Concluding remarks

We discussed the various asymptotic behaviors of the solutions of shell models, and the numerical difficulties that need to be addressed when seeking an effective and reliable finite element procedure for shell structures in the framework of such complex physical behaviors.

We also summarized some results and pointed out open problems pertaining to the analysis of shells in engineering practice, and in particular regarding general shell elements and 3D-shell elements. Although these elements were primarily designed to fulfill some applicative specifications, they can be mathematically analyzed and made as reliable as other shell procedures, with additional benefits.

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