Open problems in elasticity

J.M. Ball*

Mathematical Institute, University of Oxford, Oxford OX1 3LB, UK

Abstract

Some outstanding open problems of nonlinear elasticity are described.

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1. Introduction

This paper highlights various fundamental open problems in three-dimensional nonlinear elasticity. As a rough statement one can say that almost every fundamental question in the theory is open! This is a corollary of a general lack of understanding of systems of quasilinear partial differential equations, both static and dynamic. A detailed survey, with references, of these open problems, and others, can be found in Ball [1]. Further open questions related to the numerical computation of solutions are discussed in Ball [2].

Consider an elastic body occupying in a reference configuration a bounded domain $\Omega \subset \mathbf{R}^3$ with Lipschitz boundary $\partial\Omega$. For simplicity we suppose that the body is homogeneous with stored-energy function $W = W(\mathbf{A})$, in terms of which the elastic energy of a deformation $\mathbf{y}: \Omega \to \mathbf{R}^3$ is given by

$$I(\mathbf{y}) = \int_\Omega W(D\mathbf{y}(x))\, dx \tag{1}$$

We denote by $M_+^{3\times3}$ the set of real 3×3 matrices $\mathbf{A}$ with $\det \mathbf{A} > 0$. We suppose that $W: M_+^{3\times3} \to (0, \infty)$ is smooth, bounded below, and satisfies

$$W(\mathbf{A}) \to \infty \text{ as } \det \mathbf{A} \to 0+ \tag{2}$$

and the *frame-indifference* condition

$$W(\mathbf{RA}) = W(\mathbf{A}) \quad \text{for all } \mathbf{R} \in SO(3),\ \mathbf{A} \in M_+^{3\times3} \tag{3}$$

Again, for simplicity, we suppose that there is no body force, and that the body is subjected to mixed boundary conditions given by $\mathbf{y}|_{\partial\Omega_1} = \bar{\mathbf{y}}$, where $\partial\Omega_1$ is a relatively open subset of $\partial\Omega$ and $\bar{\mathbf{y}}: \partial\Omega_1 \to \mathbf{R}^3$ is given, with the remainder of the boundary traction-free.

* Tel.: +44 1865 273577; Fax: +44 1865 273 586;
E-mail: ball@maths.ox.ac.uk

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2. Elastostatics

For elastostatics the equilibrium equations are the Euler-Lagrange equations for the functional (1), namely

$$\operatorname{div} \mathbf{T}_R(D\mathbf{y}) = 0 \tag{4}$$

where $\mathbf{T}_R(\mathbf{A}) = D_\mathbf{A} W(\mathbf{A})$ is the *Piola-Kirchhoff stress*.

There are two main approaches to the existence of solutions to Eq. (4). The first is via the direct method of the calculus of variations. It is known that the central convexity condition of the multi-dimensional calculus of variations is *quasiconvexity* in the sense of Morrey [3], and that under suitable growth hypotheses quasiconvexity is sufficient and almost necessary for the existence of at least one global minimizer $\mathbf{y}^*$ of I. However, as it stands this existence result does not apply to elasticity because of the singular behaviour (2). Instead, existence is customarily proved under the stronger hypothesis of *polyconvexity*. Little is known about the corresponding minimizer $\mathbf{y}^*$. For example, for no W is it known whether in general $\mathbf{y}^*$ is smooth, or even smooth enough to satisfy the usual weak form of Eq. (4), although other weak forms can be established (see [1]). Further, it is not known whether $\det D\mathbf{y}^*(x) \geq \epsilon$ in Ω for some $\epsilon > 0$. Although widely used models of rubber are polyconvex, little is known about verifying polyconvexity or quasiconvexity for anisotropic materials. In general there is no known tractable way of telling whether a given function is quasiconvex or not, and no local characterization is possible [4]. Even though the stored-energy functions corresponding to elastic crystals are not quasiconvex, and in general there does not exist an energy-minimizing configuration, quasiconvexity plays a key role in the understanding of the microstructure of such materials. One reason is that the quasiconvexified stored-energy function W^{qc} is that appropriate for describing the material at length-scales much greater than those of the microstructure.

The second main approach to existence is via the implicit function theorem in an appropriate Banach space (see [5]). By its nature this approach is restricted to solutions close to a given one, for example with slightly different boundary conditions. Because of the quasilinear nature of Eq. (4) it is only possible to carry it out in spaces of functions with second derivatives. Furthermore, the lack of regularity up to the boundary for the corresponding linearized equations when $\partial\Omega_1$ has a common boundary with $\partial\Omega\backslash\partial\Omega_1$ means that it is not obvious how to apply the method for typical mixed boundary conditions. In particular, there is no appropriate bifurcation theory that can be shown to apply to elasticity with mixed boundary conditions, and thus no complete three-dimensional theory of classical problems such as buckling of a rod.

3. Elastodynamics

For pure elastodynamics, the governing equations are

$$\rho_R \mathbf{y}_{tt} = \text{div}\ \mathbf{T}_R(D\mathbf{y}) \quad (5)$$

where ρ_R is the density in the reference configuration. This is a system of conservation laws in three space dimensions, and very little is known about such systems. In particular there is no theory of global existence or uniqueness of solutions to initial-boundary value problems. Even when viscoelastic damping is present, so that the equations of motion become

$$\rho_R \mathbf{y}_{tt} = \text{div}\ \mathbf{T}_R(D\mathbf{y}) + \text{div}\ \Sigma(D\mathbf{y}, D\mathbf{y}_t) \quad (6)$$

there is no satisfactory theory. In order to be frame-indifferent, the viscous stress Σ must have the form

$$\Sigma(D\mathbf{y}, D\mathbf{y}_t) = D\mathbf{y}\Sigma(\mathbf{U}, \mathbf{U}_t) \quad (7)$$

where $\mathbf{U} = (D\mathbf{y}^T D\mathbf{y})^{\frac{1}{2}}$. This makes the viscous term in Eq. (6) genuinely quasilinear. Here we have not considered thermal effects; the corresponding systems of governing equations are even more intractable.

The lack of a good mathematical theory of dynamics means that we are unable to bring to bear on elasticity the apparatus of dynamical systems theory, and in particular to properly study the relationship between statics and dynamics, or the stability of equilibrium or other solutions.

4. Current advances

Nevertheless fundamental advances in understanding are being made, as nonlinear analysis brings its power to bear on elasticity. Thus we see interesting attempts to establish the status of elasticity with respect to atomistic theory (see, for example, [6–9]), the beginnings of a rigorous theory of the relationship between three-dimensional elasticity and theories of rods, plates and shells, (see, for example, [10–12], the first steps towards a mathematical understanding of fracture mechanics (see, for example, [13,14,15] and the recognition that elasticity has much to say about crystal microstructure and other defects in crystals (see, for example, [16–19]). At the same time, as has happened throughout its history, elasticity is stimulating the growth of new mathematics; an interesting example is the study by Chlebík and Kirchheim [20], and Kirchheim and Preiss [21] of mappings whose gradient takes only a finite number of values A_i, $i = 1, \ldots, N$, where rank $(A_i - A_j) > 1$ for $i \neq j$.

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