# The shear-layer instability of a circular cylinder wake

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A reinterpretation is made of previously published data concerning the frequency of the instability waves in the separated shear layer from a circular cylinder for Reynolds numbers in the range $10^3$–$10^5$. An accurate fit to the observed variation can be achieved using a piecewise fit based on theoretical and empirical arguments. A logical conclusion is that the ratio of the frequency of the instability waves to the Kármán vortex shedding frequency is indeed determined by the *boundary-layer* properties at separation, as suggested by Bloor. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1852581]

Bloor[1] made possibly the earliest systematic study of instability waves in the separated shear layer from a circular cylinder. This instability has subsequently become known as the Bloor–Gerrard instability. The occurrence of these shear-layer vortices is now well established. They have been beautifully visualized by Wei and Smith,[2] Kourta *et al.*,[3] and Prasad and Williamson,[4] amongst others.

By assuming that the instability occurring in the separating shear layers was governed by boundary-layer properties at separation, she used simple boundary-layer theory to deduce the relationship between the frequency of these waves $f_{SL}$, the Kármán vortex frequency $f_K$, and the Reynolds number $Re$, to be $f_{SL}/f_K \propto Re^{1/2}$, where $Re = U_\infty D/\nu$, and $U_\infty$ is the free-stream velocity, $D$ is the cylinder diameter, and $\nu$ is the kinematic viscosity. This theoretical relationship was supported by her experimental data.

Two decades later, Wei and Smith[2] used a vortex-counting technique in conjunction with their flow visualizations to find that the shear-layer vortex frequency varied according to the relationship $f_{SL}/f_K = 0.0047Re^{0.87}$; markedly different from Bloor's.[1] To explain the difference between their results and Bloor's[1] results, they suggested that it is more appropriate to assume that the shear-layer instability scales by local conditions within the shear layer. In particular, they proposed that the momentum thickness in the middle of the exponential growth region of the separated shear layer should be the appropriate lengthscale for scaling, rather than the attached boundary-layer thickness at separation.

Only slightly later, Kourta *et al.*[3] presented power spectra of signals from a hot wire located in the near wake of the circular cylinder in the $Re$ range 2000–16 000. Analysis of these hot-wire frequencies supports the $Re^{1/2}$ prediction of Bloor. It has been speculated that the hydrogen bubbles used in the experiments of Wei and Smith[2] may have artificially disturbed the shear layers, e.g., see Zdravkovich.[5]

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Since the mid-1990s, the situation appears to have changed once again. In several recent studies, the conclusion appears to be that the experimental data are best fitted by a functional relationship approximately halfway between that of Bloor[1] and Wei and Smith.[2] In particular, Prasad and Williamson[4] re-examined previous data sets together with their own and concluded that the best-fit exponent for the $Re$ dependence is 0.69. In addition, Norberg[6] independently analyzed all the available data in this Reynolds number range and estimated a similar value of 0.68. Prasad and Williamson[4] went further by proposing a theoretical explanation for the observed dependence basing the frequency selection on the length and velocity scales of the shear layer at the variable downstream position where the shear-layer signal can first be detected experimentally. Using available data on how this position depends on Reynolds number, and including corrections due to the Strouhal number and base pressure coefficient variations, they were able to estimate a theoretical value for the exponent of 0.67.

In this paper we propose a different interpretation by re-examining the data obtained by Prasad and Williamson[4] and Norberg.[7] We focus on these two data sets because they represent experiments undertaken with great care producing consistent data with very little scatter. Importantly, these data sets are in good agreement in the region of overlap.

Figure 1(a) shows the frequency variation of the shear-layer vortices over the Reynolds number range $10^3 < Re < 10^5$, from the experimental measurements of Prasad and Williamson[4] and Norberg.[7] The line of best fit to the combined data set is $f_{SL}/f_K = 0.021Re^{0.69}$, as given by Prasad and Williamson[4] and independently verified by ourselves.

A close examination of the data clearly suggests that instead of a universal fit applying over the entire Reynolds number range, a more accurate characterization consists of separating the data into discrete intervals that can be fitted independently. Figure 1(b) shows the two longest intervals where the data appears to be closely linear. These intervals cover the approximate Reynolds number ranges $1500 \leqslant Re \leqslant 5000$ (range 1) and $10000 \leqslant Re \leqslant 50000$ (range 2). For both ranges, the best fit to the data gives a considerably lower Reynolds number exponent than the global exponent.

 

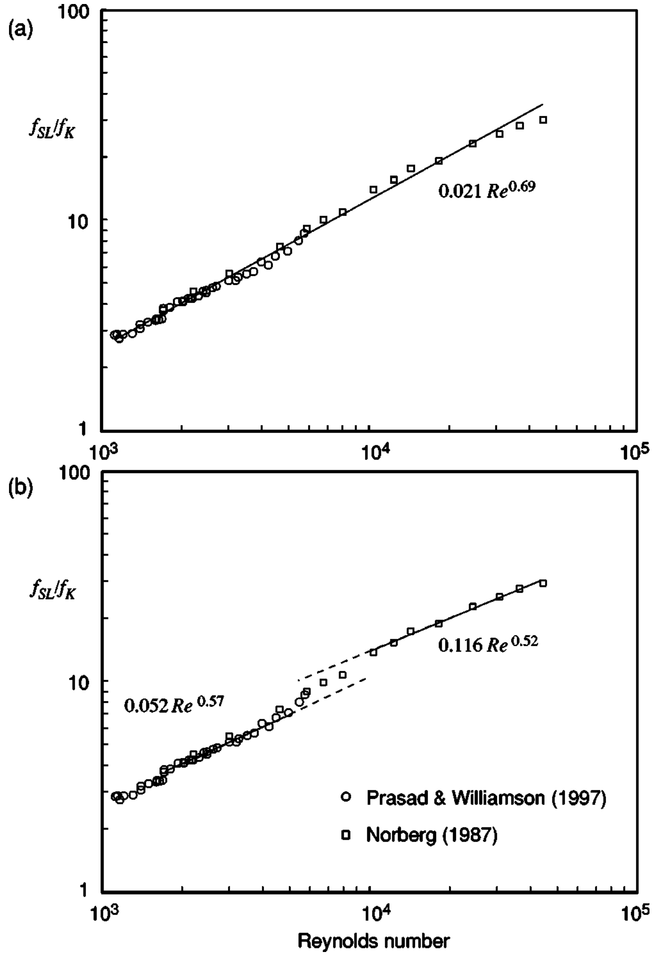


FIG. 1. (a) Variation of the shear-layer frequency ratio with Reynolds number from the studies of Prasad and Williamson (Ref. 4) and Norberg (Ref. 7). The line of best fit to the data is shown. (b) Proposed alternative fit to the data.

The dependence is $f_{SL}/f_K \propto Re^{0.57\pm0.04}$ for range 1 and $Re^{0.52\pm0.06}$ for range 2. There is also some evidence for a short linear regime for $Re<1500$, but we focus on the two main ranges in the following discussion.

The error bounds are statistical error estimates based on a 95% confidence interval assuming that at each Reynolds number the fractional error in the measured frequency ratio is normally distributed. Importantly, note that the exponent ranges defined by these error bounds *do not* include the best-fit global exponent of 0.69, calculated for the entire data set. In fact, an exponent of 0.69 is five or more standard deviations outside the best fit value for each of the individual fits. The most likely explanations are either that both sets of data contain some systematic error, or that the proposal that a universal exponent applies over the entire range is incorrect. Hence we ask the obvious question: Does it make physical sense to consider separate ranges? We feel the answer is definitely yes.

Figure 2 shows how three related flow parameters vary over the relevant Reynolds number range. Figures 2(a) and 2(b) are reproduced from Norberg's[7,8] papers on circular cylinder wakes. These figures show the variation in Strouhal number, *St*, and base pressure coefficient, $C_{Pb}$, with Reynolds

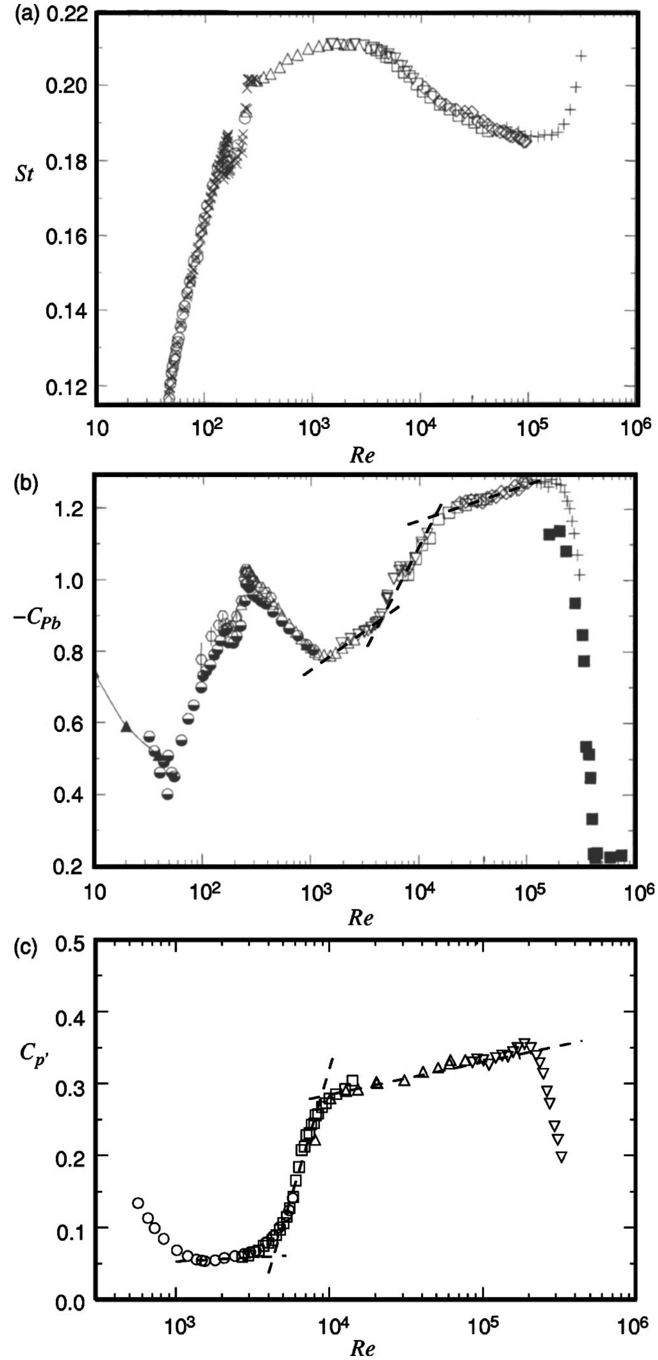


FIG. 2. (a) Strouhal number, (b) base suction coefficient, and (c) fluctuating lift coefficient at 90°, as functions of Reynolds number. These plots have been reconstructed from Fig. 3 of Norberg (Refs. 7 and 8) and Fig. 6 of Norberg (Ref. 9). The overlaid dashed line segment appearing in the last plot highlights the relatively constant behavior within ranges 1 and 2.

number. Figure 2(c) shows the Reynolds number variation of the fluctuating pressure at $\theta=90^\circ$, close to the separation point, from Norberg's[9] paper. First, consider the behavior of the base pressure coefficient. In terms of the range of interest here, $C_{Pb}$ varies relatively slowly over ranges 1 and 2 identified and considerably more rapidly in between as indicated by the overlaid linear segments. This effect is shown even more strongly in the plot of the fluctuating lift. The latter shows two distinct Reynolds number ranges where the value is approximately constant corresponding remarkably closely

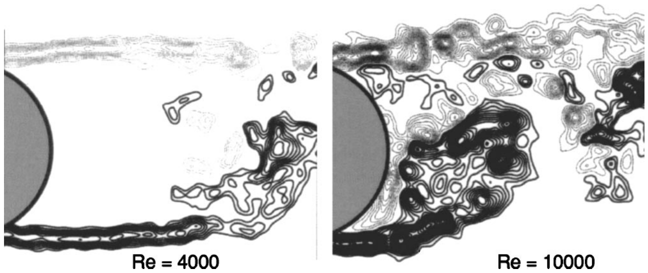


FIG. 3. Typical instantaneous vorticity fields showing the change in the near-wake structure as the Reynolds number is varied. These images have been obtained from PIV measurements obtained by Saelim and Rockwell (Ref. 10).

to the ranges given above. These approximate ranges are $1000 < Re < 4000$ and $Re > 10000$. In between, the fluctuating pressure changes rapidly with Reynolds number. Physically, the observed variation can be associated with the dramatic shortening of the mean separation bubble from large values at low Reynolds numbers to small values at high Reynolds numbers. For $Re \geqslant 10^4$ the variation in bubble length, or formation length, is considerably less.

The remarkable difference in the immediate wake of the cylinder as the Reynolds number is varied is demonstrated clearly in Fig. 3. These images show typical instantaneous vorticity fields obtained by PIV measurements. They have been provided by Saelim and Rockwell.[10] They show the wake at $Re \leqslant 4000$, where the formation length is relatively long, and the severely shortened wake at $Re = 10\,000$. The transition from the long formation length, low Reynolds number wake state to the short, high Reynolds number wake state takes place over the Reynolds number range $4000 \leqslant Re \leqslant 10\,000$, consistent with the marked increase in the fluctuating lift at 90° over this interval as shown in Fig. 2(c). This behavior was documented in the pioneering studies of Linke.[11] When the formation length is long, the separating shear layers are relatively unaffected by the Karman shedding, which occurs much further downstream. On the other hand, when the formation length is short at the higher Reynolds numbers, the separating shear layers are strongly affected. In particular, the shear layers *flap* considerably, and they will be stretched more and hence will be thinned, due to the influence of the forming Karman vortices, which are in close proximity. Thus, we are left with the situation in which the shear layers and shear-layer environment are approximately similar separately in regimes 1 and 2 (except for the natural shear-layer thinning with Reynolds number) but not in between. In addition, if the shear layers are stretched longitudinally through the action of the forming Karman vortices, a readjustment to a higher frequency ratio above the underlying trend should be observed, due to the increased narrowing of the shearlayers. This is indeed what is suggested by Fig. 1(b) above. To reiterate, given that the shear layer changes remarkably in between ranges 1 and 2, it is not surprising that the hypothetical dependence proposed by Bloor breaks down since it depends on the shear layer maintaining self-similarity.

To this point we have argued that the relationship proposed by Bloor should apply over the approximate ranges 1 and 2, but not in between. However, the best fit Reynolds number indices are still slightly higher than the theoretical value suggested by Bloor. Why is it so?

To address this issue we return to the analysis of Bloor[1] who proposed that the shear-layer frequency should scale as $f_{SL} \propto U_{bl}/\delta_s$, where $U_{bl}$ is the velocity at the edge of the boundary layer and $\delta_s$ is the boundary layer thickness at the point of separation. Neither the boundary layer velocity or thickness appear to have been measured systematically over the Reynolds number range $10^3 < Re < 10^5$, although some point measurements exist. However, for sufficiently high Reynolds number, the velocity at the edge of the boundary layer is directly dependent on the pressure coefficient there, $C_{P_{bl}}$. While the variation of this parameter also has not been documented in the literature, it should be approximately equal to the base pressure coefficient, since the pressure remains almost constant in the separation zone at the back of the cylinder. This has been pointed out previously by Williamson, Wu and Sheridan,[12] and Roshko;[13] indeed Prasad and Williamson[4] assume this association in their derivation of the shear-layer frequency ratio variation. Specifically,

$$U_{bl} \approx U_\infty (1 - C_{Pb})^{1/2}.$$

For a laminar flat-plate boundary layer, *at a fixed point* the boundary-layer thickness scales as $\delta/D \propto Re^{-1/2}$. However, for a circular cylinder, the situation is a little more complicated. The separation point is not fixed. It moves from the rear of the cylinder at low Reynolds numbers towards the front at higher Reynolds numbers. It is difficult to find definitive data on the exact variation with Reynolds number, especially since the separation point moves considerably during a shedding cycle. From collected data from a number of authors presented in Zdravkovich,[5] the separation angle is about 95° at $Re = 300$, dropping to 82° at 12 000. Its value reduces only very slowly, if at all, for higher Reynolds numbers until the onset of the drag crisis at $Re \simeq 2 \times 10^5$. Because of the considerable variation of the separation point, especially over range 1, the boundary layer thickness at separation will deviate from the $Re^{-1/2}$ law. It is expected that the boundary layer thickness should obey a relationship of the form

$$\delta_s/D \propto Re^{-1/2} f(\theta_s(Re)),$$

where $f(\theta_s)$ accounts for the variation due to the movement of the separation angle, $\theta_s$, with Reynolds number.

Putting these relationships together allows the frequency ratio to be written

$$\frac{f_{SL}}{f_K} = \kappa \frac{U_\infty}{D f_K} (1 - C_{Pb})^{1/2} \frac{Re^{1/2}}{f(\theta_s)} = \kappa \left[ \frac{(1 - C_{Pb})^{1/2}}{St f(\theta_s)} \right] Re^{1/2},$$

where $St = f_K D / U_\infty$ is the Strouhal number and $\kappa$ is a proportionality constant. Thus, the relationship proposed by Bloor requires that the base pressure coefficient, Strouhal number, and the separation angle do not vary significantly over the Reynolds number range of interest. While the base pressure coefficient and Strouhal number can be obtained from Fig. 2, the function $f(\theta_s)$, or equivalently $\delta_s(Re)$, is not readily available.

This term was estimated numerically by solving the steady Navier–Stokes equations directly for laminar flow past a cylinder with a symmetry boundary condition along the centerline. An extensively tested third-order finite-element code was used. A grid resolution study was performed to ensure the results were grid independent. In support of this approach, Dimopoulos and Hanratty[14] showed experimentally that the separation angle for a steady flow produced with the aid of a splitter plate matched the time-mean separation angle for the flow without a splitter plate, although this was for much lower Reynolds numbers ($Re < 300$). The separation point and boundary layer thickness were measured directly from the results. At $Re=1500$ the separation point was measured as $\theta_s \simeq 91°$, dropping to $\theta_s \simeq 84°$ at $Re=5000$. These values appear to be consistent with the experimental values given above. A power-law fit for range 1 gave $\delta_s/D \propto Re^{-0.528}$. Thus, $f(\theta_s(Re)) \propto Re^{-0.028}$. The separation angle changes more slowly over range 2, hence the variation of $f$ can probably be neglected for that range.

Prasad and Williamson[4] estimated the contribution of the Strouhal number and base pressure variation over the entire Reynolds number range as

$$\frac{(1-C_{Pb})^{1/2}}{St} \propto Re^{0.075}.$$

(See Fig. 16 from that paper.) In fact, this is an overestimate of the variation for ranges 1 and 2 individually because the base pressure gradient in each of these ranges is less, as seen from Fig. 2. We performed our own fits to find the correction factor (including the new contribution from the changing separation angle) is

$$\left[\frac{(1-C_{Pb})^{1/2}}{Stf(\theta_s)}\right] \propto Re^{0.084} \text{ and } \left[\frac{(1-C_{Pb})^{1/2}}{Stf(\theta_s)}\right] \propto Re^{0.046},$$

for ranges 1 and 2, respectively.

Therefore, if the boundary-layer properties at separation govern the shear-layer instability then the predicted frequency variation should be $f_{SL}/f_K \propto Re^{0.584}$ for range 1 and $f_{SL}/f_K \propto Re^{0.546}$ for range 2. These agree remarkable well with the measured variations of $Re^{0.57\pm0.04}$ and $Re^{0.52\pm0.06}$, respectively.

Note that the correction due to the change in the position of the separation angle is very small and neglecting it does not affect the conclusion. Also note that the Strouhal number and base pressure coefficient corrections are well established and have been used by Prasad and Williamson[4] in formulating their theoretical $f_{SL}/f_K \propto Re^{0.69}$ relationship. The main difference from their analysis is that they use a different lengthscale to derive the frequency ratio. They assume that the appropriate length scale is the shear-layer thickness at a downstream distance at which the shear-layer fluctuations can first be sensed experimentally. This seems somewhat artificial, since it may depend on the sensitivity of the experimental measuring equipment. Finally, note that the small linear regime in the range $1200 \leqslant Re \leqslant 1500$ can be incorporated into range 1 within the error bounds.

Thus, the current analysis suggests that, to within experimental uncertainty, the boundary-layer properties at separation are sufficient to account for the observed Reynolds number variation of the shear-layer frequency, as proposed by Bloor.[1]


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